Probability Theory

Solution sheet 12

Solution 12.1

(a) Since $X_0 = a \in [0,1]$, from the assumption that $P[X_{n+1} = \frac{X_n}{2} \text{ or } X_{n+1} = \frac{1+X_n}{2}] = 1$ we can use induction argument to conclude that $0 \leq X_n \leq 1$ for all n, Hence each X_n is integrable. Moreover, it holds that

$$\begin{aligned}
& E[X_{n+1}|\mathcal{F}_n] \\
= & \frac{X_n}{2} P[X_{n+1} = \frac{X_n}{2}|\mathcal{F}_n] + \frac{1+X_n}{2} P[X_{n+1} = \frac{1+X_n}{2}|\mathcal{F}_n] \\
= & \frac{X_n}{2}(1-X_n) + \frac{1+X_n}{2} X_n = X_n.
\end{aligned}$$

Thus X_n is a non-negative martingale, and by the Martingale Convergence Theorem, X_n converge to a random variable X_∞ a.s. Besides, we have that X_n is bounded by 1, then the convergence holds also in L^p for all $p \geq 1$ due to the Dominated Convergence Theorem.

(b) We have that

$$E[(X_{n+1} - X_n)^2] = E[E[(X_{n+1} - X_n)^2|\mathcal{F}_n]] \tag{1}$$

$$= E[E[(X_{n+1}^2 - 2X_{n+1}X_n + X_n^2|\mathcal{F}_n]].$$

It is easy to see that

$$\begin{aligned}
E[X_{n+1}^2|\mathcal{F}_n] &= \left(\frac{X_n}{2}\right)^2 P[X_{n+1} = \frac{X_n}{2}|\mathcal{F}_n] + \left(\frac{1+X_n}{2}\right)^2 P[X_{n+1} = \frac{1+X_n}{2}|\mathcal{F}_n] \\
&= \left(\frac{X_n}{2}\right)^2 (1-X_n) + \left(\frac{1+X_n}{2}\right)^2 X_n = \frac{X_n}{4}(1+3X_n).
\end{aligned}$$

Plugging this in (1) we have that

$$E[E[(X_{n+1} - X_n)^2|\mathcal{F}_n]] = E[\frac{X_n}{4}(1+3X_n) - 2X_n^2 + X_n^2] = \frac{1}{4}E[X_n(1-X_n)].$$

Solution 12.2

(a) Since $X_n \xrightarrow{n\to\infty} X$ in distribution, we know by Proposition 2.7, p. 50 of the lecture notes that one can construct random variables Y_n, $n \in \mathbb{N}$, and Y on a common probability space $(\Omega', \mathcal{A}', P')$, such that $Y_n \stackrel{d}{=} X_n$, for all $n \in \mathbb{N}$, $Y \stackrel{d}{=} X$, and $Y_n \to Y$, P'-almost surely. It is easy to verify that the family $\{Y_n\}_{n\in\mathbb{N}}$ is also uniformly integrable, since

$$\lim_{M\to\infty} \sup_{n\in\mathbb{N}} E_{P'}\left[|Y_n| 1_{\{|Y_n|>M\}}\right] \stackrel{Y_n \stackrel{d}{=} X_n}{=} \lim_{M\to\infty} \sup_{n\in\mathbb{N}} E_P\left[|X_n| 1_{\{|X_n|>M\}}\right] = 0.$$

So by (3.6.18)-(3.6.19), p. 112 of the lecture notes, we have

$$E_{P'}[Y_n] \xrightarrow{n\to\infty} E_{P'}[Y],$$

and the result follows since $E_P[X_n] = E_{P'}[Y_n]$, for all $n \in \mathbb{N}$, and $E_P[X] = E_{P'}[Y]$.

(b) Since convergence in probability implies convergence in distribution, by using Fatou's Lemma, we can find an M large enough such that

$$E_P[|X|1_{\{|X|>M\}}] = E_{P'}[|Y|1_{\{|Y|>M\}}] \overset{\text{Fatou}}{\leq} \liminf E_{P'}\left[|Y_n|1_{\{|Y_n|\geq M\}}\right] \leq \epsilon. \tag{2}$$

Moreover, by convergence in probability, there exists $n_0 \geq 0$ such that, for all $n \geq n_0$, we have

$$P\Big[\underbrace{|X_n - X| \geq \epsilon}_{A_n}\Big] < \frac{\epsilon}{M}.$$

Hence, for all $n \geq n_0$, by (3.6.21) in lecture notes it holds that

$$\begin{aligned} E_P[|X_n - X|] &\leq E_P\left[|X_n - X|1_{\{|X_n|\leq M, |X|\leq M\}}\right] \\ &\quad + 3\underbrace{E_P\left[|X_n|1_{\{|X_n|>M\}}\right]}_{\leq\epsilon} + 3\underbrace{E_P\left[|X|1_{\{|X|>M\}}\right]}_{\leq\epsilon} \\ &\leq E_P\Big[\underbrace{|X_n - X|1_{\{|X_n|\leq M, |X|\leq M\}}}_{\leq 2M} 1_{A_n}\Big] \\ &\quad + E_P\Big[\underbrace{|X_n - X|1_{\{|X_n|\leq M, |X|\leq M\}}}_{\leq\epsilon} 1_{A_n^c}\Big] + 6\epsilon \\ &\leq 2MP[A_n] + 7\epsilon \leq 9\epsilon. \end{aligned}$$

Therefore, X_n converges to X in L^1.

Solution 12.3 Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. We need to check for all $n, k \geq 0$, $f : E^{k+1} \to \mathbb{R}$ bounded and measurable, there exists a (measurable) function $h : E \to \mathbb{R}$ (only depending on f) such that for *all* $n \geq 0$,

$$E[f(X_n, \cdots, X_{n+k})|\mathcal{F}_n] = h(X_n)$$

(in particular, $E[f(X_n, \cdots, X_{n+k})|\mathcal{F}_n]$ is $\sigma(X_n)$ measurable). This result illustrates the "time homogeneity". We will prove it for the simple case where $f(x_0, x_1, \cdots, x_k) = 1_B(x_k)$ where $B \subset E$ and $k = 1$, the general case can be derived similarly by using the measure–theoretic induction (cf. the proof of Proposition 1.13).

Indeed, it holds that

$$E[1_{\{X_{n+1}\in B\}}|\mathcal{F}_n] = E[1_{\{\Phi(X_n, Y_{n+1})\in B\}}|\mathcal{F}_n] = \Psi_B(X_n), \tag{3}$$

where

$$\Psi_B(x) = E[1_{\{\Phi(x, Y_{n+1})\in B\}}] = P[\Phi(x, Y_{n+1}) \in B],$$

and the last equality in (3) follows from the fact that Y_{n+1} is independent of $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, and that X_n is $\mathcal{F}_n$-measurable. (It is an easy exercise to check that

$$E[1_{\{\Phi(X_n, Y_{n+1})\in B\}}|\mathcal{F}_n]1_{\{X_n=x\}} = P[\Phi(x, Y_{n+1}) \in B]1_{\{X_n=x\}}$$

for all $x \in E$). This shows that $(X_n)_{n\geq 0}$ is a time homogenous Markov chain and the transition matrix is given through

$$Q(x, y) = P_x[X_1 = y] = E_x\Big[E[1_{\{X_1=y\}}|\mathcal{F}_0]\Big] = P[\Phi(x, Y_1) = y],$$

for $P_x[X_0 = x] = 1$ (we may take $P_x = P[\cdot|X_0 = x]$). Here we remark that the time homogeneity of $(X_n)_{n\geq 0}$ follows from the above observation that for all $n \geq 0$, $x, y \in E$,

$$P[X_{n+1} = y|X_n = x] = P[X_1 = y|X_0 = x] = Q(x, y).$$