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Probability Theory

Solution sheet 13

Solution 13.1 Let $x \in A$, $\tau_A = 0$ P_x-a.s. For $x \in A^c$, we have that for all $k \geq 0$,

$$P_x(\tau_A > (k+1)n) \leq P_x(\tau_A > kn, X_{(k+1)n} \in A^c) = E_x[E_x[1_{\{\tau_A > kn\}} 1_{\{X_{(k+1)n} \in A^c\}} | \mathcal{F}_{nk}]]$$
$$\overset{\text{Markov}}{=} E_x[1_{\{\tau_A > kn\}} \underbrace{P_{X_{kn}}[X_n \in A^c]}_{\leq 1-\alpha}] \leq (1-\alpha) P_x(\tau_A > kn).$$

From the last we get by induction that $P_x(\tau_A > kn) \leq (1-\alpha)^k$ and taking the limit as k goes to infinity we get that,

$$P_x(\tau_A = +\infty) = \lim_{k \to \infty} P_x(\tau_A > kn) = 0.$$

Solution 13.2 By (4.2.58) in lecture notes, the process

$$M_n := f(X_n) - \sum_{k=0}^{n-1} (Qf - f)(X_k) \tag{1}$$

for $n \geq 1$, $M_0 = 0$ is an $\mathcal{F}_n$-martingale under P_x. Note that here Qf coincides with Kf appeared in (4.2.58) when the state space E is countable.
Furthermore, it is not hard to check that τ_F is an $\mathcal{F}_n$–stopping time (as F is a countable set), and therefore by Corollary 3.24 (optional stopping theorem) the stopped process $M_{n \wedge \tau_F}$ is also a martingale under P_x. Now we consider the stopped process $f(X_{n \wedge \tau_F})$. It holds that (see (1))

$$f(X_{n \wedge \tau_F}) = M_{n \wedge \tau_F} + \sum_{k=0}^{(n \wedge \tau_F)-1} (Qf - f)(X_k). \tag{2}$$

Moreover, since $X_k(\omega) \in F^c$ for all $k < \tau_F(\omega)$ and all $\omega \in \Omega$, the assumption that $f(x) \geq Qf(x)$ for all $x \in F^c$ ensures that the process $\sum_{k=0}^{(n \wedge \tau_F)-1} (Qf - f)(X_k)$, $n \geq 0$ is non–increasing. In view of (2) we can conclude that $f(X_{n \wedge \tau_F})$ is a supermartingale under P_x, cf. Proposition 3.19. The same argument shows that it is a martingale if $Qf(x) = f(x)$ for all $x \in F^c$,

Solution 13.3

(a) For $x \in \mathbb{Z}$ one has

$$b(x) = E_x[X_1] = \sum_{y \in \mathbb{Z}} y Q(x,y),$$
$$a(x) = E_x[(X_1 - b(x))^2] = \sum_{y \in \mathbb{Z}} (y - b(x))^2 Q(x,y).$$

(b) Using the Markov property we have (recall that ϑ_n is the shift operator on Ω, see p. 147)

$$E_x[X_{n+1}] = E_x[X_1 \circ \vartheta_n] = E_x[E_{X_n}[X_1]] = E_x[b(X_n)],$$
$$E_x[X_{n+1}^2] = E_x[X_1^2 \circ \vartheta_n]$$
$$= E_x[E_{X_n}[X_1^2]] = E_x[\mathrm{Var}_{X_n}(X_1) + E_{X_n}[X_1]^2] = E_x[a(X_n) + b(X_n)^2].$$

Hence we have

$$\begin{aligned}
\mathrm{Var}_x(X_{n+1}) &= E_x[X_{n+1}^2] - E_x[X_{n+1}]^2 \\
&= E_x[a(X_n) + b(X_n)^2] - E_x[b(X_n)]^2 \\
&= E_x[a(X_n) + b(X_n)^2] - \Big(E_x[b(X_n)^2] - \mathrm{Var}_x(b(X_n))\Big) \\
&= \mathrm{Var}_x(b(X_n)) + E_x[a(X_n)].
\end{aligned}$$