ISING MODEL

INTRODUCTION

1 STATISTICAL MECHANICS

General idea: Study physical systems with a very large number of elements using tools from probability theory.

Examples:
- population dynamics ($\simeq 10^9$ individuals)
- a glass of water ($\gg 10^{23}$ molecules)
- a piece of Iron ($\gg 10^{23}$ atoms)
- cars in a high way
- a forest of trees.
- a porous stone

...

Here comes probability theory:

Giving an exact description of such system is very hard (eg. for water, one needs to understand $\gg 10^{23}$ equations!) Instead, we give a probabilistic description. Each element has a random behaviour, and the system is described by very few parameters.

We are interested in the large-scale behaviour of such system.

Examples:
- population dynamics: survival / extinction ?
- water : solid / liquid / gaz ?
- Iron : paramagnetic / ferromagnetic ?

...

For such systems, we often observe a sharp phase transition: a small change in the parameters may give rise to completely different macroscopic behaviours (think of water at 0°C).

<u>Modelling</u>

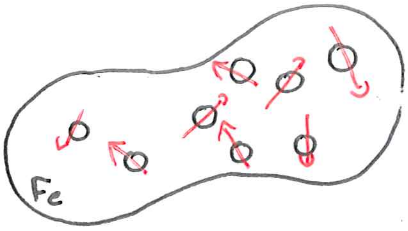
- Ω = {"possible states for the system"}
- P_β = probability measure on Ω, indexed by a parameter β.

In this course, we will study the <u>Ising model</u>. Initially introduced as a model for ferromagnetism, it has become one of the most important model in statistical physics with applications in various areas of science (thermodynamics, neuroscience, ...)

2. PARAMAGNETIC / FERROMAGNETIC PHASE TRANSITION

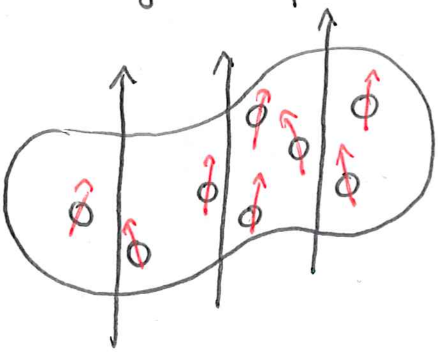
Ising model was introduced by Lenz in 1920 in view of a theoritical understanding of the para/ferromagnetic phase transition. The model was named after Ising (Lenz's student) who studied the one dimensional version of the model in his PhD thesis (1925). In this section, we give a brief description of the para/ferromagnetic phase transition.

- Consider a piece of iron a temperature T, without external field.



The iron is not magnetized.

- Add a magnetic field.

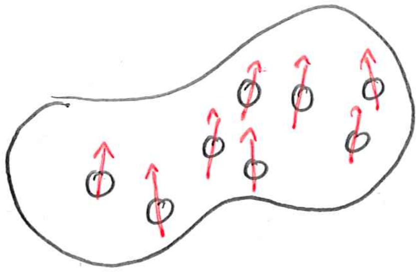


The iron gets magnetized in the same direction as the field.

- Remove the magnetic field

2 cases

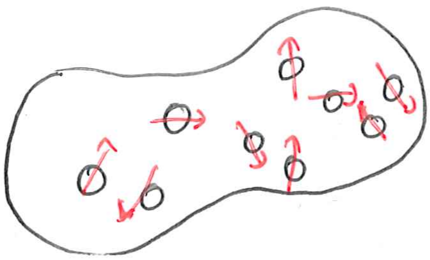
$T < T_c$



The iron remains magnetized

"Ferromagnetic behaviour".

$T > T_c$



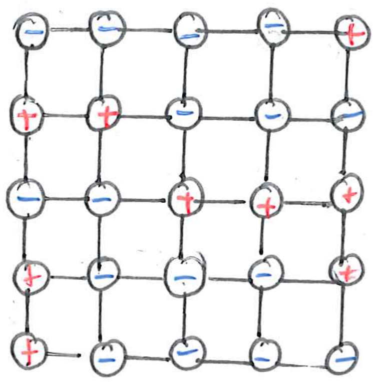
The iron loses its magnetization

"Paramagnetic behaviour"

$T_c(Fe) = 1034\,K$ Curie temperature (Pierre Curie 1895).

3 MODELISATION : BOLTZMAN FORMALISM. (dim = 2)

$\Lambda = \{-n, \dots, n\}^2$



"particles"

<u>spin configuration</u> : $\sigma = (\sigma_x)_{x \in \Lambda} \in \{-1, 1\}^\Lambda$

$\sigma_x = +1$ "spin up"

$\sigma_x = -1$ "spin down"

<u>goal</u> : define a probability measure μ_β on $\{-1,1\}^\Lambda$ which favors configuration with few neighbour disagreements ($\oplus - \ominus$ disagreement). "A particle tries to have the same spin as its neighbours". The parameter $\beta = \frac{k}{T}$ describes the strength of the interaction.

<u>Energy of a configuration $\sigma \in \{-1,1\}^\Lambda$</u>

For $\beta \geq 0$, let

$$\boxed{H_\beta(\sigma) = -\beta \sum_{x,y \text{ neighbours}} \sigma_x \sigma_y}$$

<u>Remark</u>: $H_\beta(\sigma) = \beta \sum_{x,y \text{ neigh.}} \left(2 \mathbb{1}_{\sigma_x \neq \sigma_y} - 1\right)$

↳ the energy of σ is large when the number of disagreement is large.

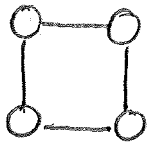
<u>Probability of a configuration</u>

$$\boxed{\mu_\beta(\sigma) = \frac{1}{Z_\beta} e^{-H_\beta(\sigma)}} \quad \text{where } Z_\beta = \sum_{\sigma \in \{\pm 1\}^\Lambda} e^{-H_\beta(\sigma)}$$

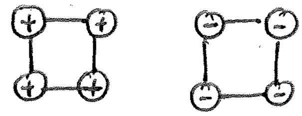
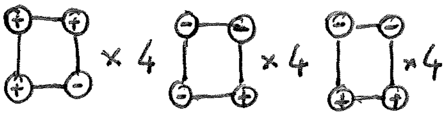
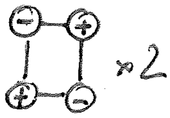
Z_β is <u>the partition function</u>. It is defined in such a way that μ_β is a probability measure.

<u>idea</u>: if σ has a "large" energy $H_\beta(\sigma)$, then $\mu_\beta(\sigma)$ is "small".

4 ISING MODEL ON A SQUARE.

$G =$  graph with vertex set $\{0,1\}^2$

In this simple case, we can compute $\mu_\beta(\sigma)$ for every spin configuration (there are 16 configurations, 2 configurations with no disagreement, 12 configurations with 2 disagreements, and 2 configurations with 4 disagreements.

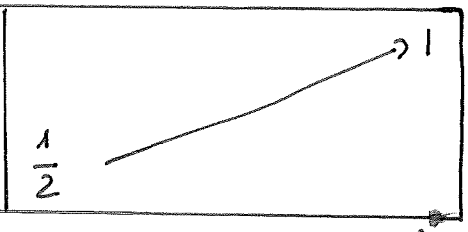
Configuration σ	Energy	Probability $\mu_\beta(\sigma)$: $\beta \searrow 0$	$0 < \beta < \infty$	$\beta \nearrow \infty$
	-4β	$\frac{1}{16}$	$\frac{1}{Z_\beta} e^{4\beta}$	$\frac{1}{2}$
	0	$\frac{1}{16}$	$\frac{1}{Z_\beta}$	0
	$+4\beta$	$\frac{1}{16}$	$\frac{1}{Z_\beta} e^{-4\beta}$	0

$$Z_\beta = 2e^{4\beta} + 12 + 2e^{-4\beta}$$

Interaction between the spins at $(0,0)$ and $(1,1)$

$\beta \to 0$ $\qquad\qquad$ $\beta \to \infty$

$$\mu_\beta\left(\sigma_{(0,0)} = \sigma_{(1,1)}\right) = \frac{2e^{4\beta} + 4 + 2e^{-4\beta}}{2e^{4\beta} + 12 + 2e^{-4\beta}}$$



spins independent $\qquad$ strong correlation.

asymptotic behaviour of the measure.

$$\mu_\beta \xrightarrow[\beta \to 0]{} \mu_0 \text{ (uniform)}$$

$$\mu_\beta \xrightarrow[\beta \to \infty]{} \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$$

5. MAGNETIZATION & PHASE TRANSITION.

Remark: Under the measure μ defined in section 3, the expected spin at 0 is

$$\langle \sigma_0 \rangle = 0$$

because $\mu(\sigma) = \mu(-\sigma)$ (spin-flip symmetry)

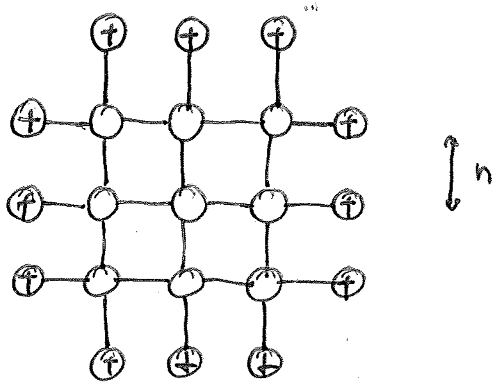
In order to break the symmetry $+/-$, we introduce the Ising measure with $+$ boundary conditions.

$\Lambda = \{-n, \dots, n\}^d$, $\partial \Lambda$ external vertex boundary in $\mathbb{Z}^d$.

We consider the spin configurations

$$\sigma : \Lambda \cup \partial \Lambda \longrightarrow \{+1, -1\}$$

such that $\sigma_{|\partial\Lambda} = +1$



and define $\mu^+_{\Lambda_n}(\sigma) = \dfrac{1}{Z^+} e^{-H^+(\sigma)}$

where $H^+(\sigma) = -\beta \sum\limits_{x,y \text{ neigh.}} \sigma_x \sigma_y$ and $Z^+ = \sum\limits_{\sigma} e^{-H^+(\sigma)}$

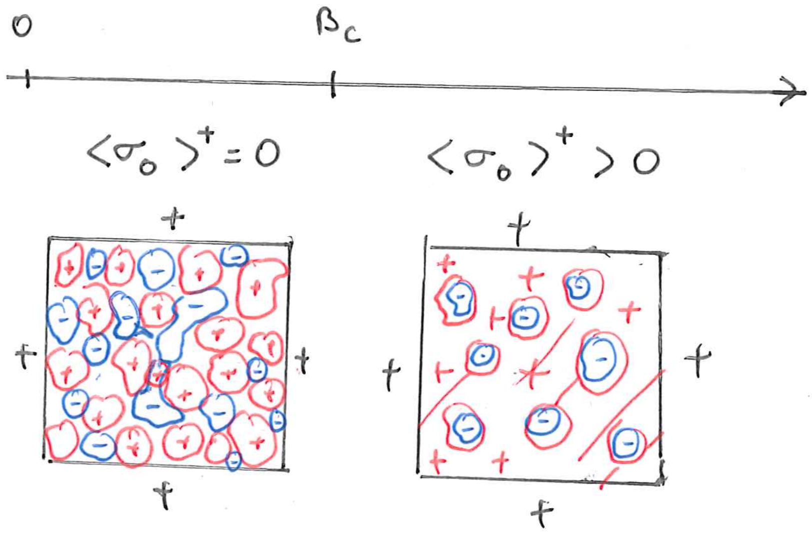
Let $\langle \sigma_0 \rangle^+_{\Lambda_n} = \mu^+_{\Lambda_n}(\sigma_0 = 1) - \mu^+_{\Lambda_n}(\sigma_0 = -1)$.

<u>Magnetization:</u>

$$\langle \sigma_0 \rangle^+ := \lim_{n \to \infty} \langle \sigma_0 \rangle^+_{\Lambda_n} .$$

(we need to prove that the limit exists).

<u>Phase transition:</u>



the origin does not "feel" the + boundary conditions

A majority of spins align on +

In the first lectures, we will prove (among other things)

- $\langle \sigma_0 \rangle^+_{\Lambda_n} \geq 0$
- $\beta_c = \infty$ in $d = 1$
- $0 < \beta_c < \infty$ in $d \geq 2$
- $\beta_c = \log(1 + \sqrt{2})$ in $d = 2$.

We will also give a more precise description of the 2 phases.

CHAPTER 1 :

ISING WITH + BOUNDARY CONDITIONS.

1 NOTATION

For $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$, we define $\|x\|_1 = \sum_{i=1}^{d} |x_i|$ (L^1 norm).

For $x, y \in \mathbb{Z}^d$, write $xy = \{x, y\}$.

graph structure: If $\|x-y\|_1 = 1$, we say that x and y are neighbours and we write $x \sim y$.

coupling constants: $J = (J_{xy})_{xy}$ s.t. $\forall x, y \in \mathbb{Z}^d$

- $J_{xy} = 0$ if $\|x-y\|_1 \neq 1$ "nearest neighbour interactions"
- $J_{xy} \geq 0$. "ferromagnetic interactions".

finite subgraphs. "finite subset"

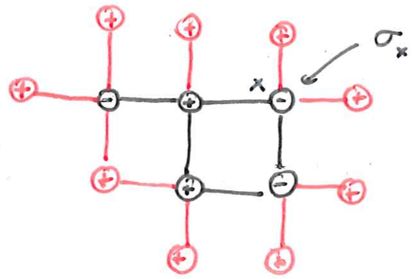
Let $\Lambda \Subset \mathbb{Z}^d$. We write

- $\partial\Lambda = \{x \in \mathbb{Z}^d \setminus \Lambda \text{ s.t. } \exists y \in \Lambda \ x \sim y\}$
- $\overline{\Lambda} = \Lambda \cup \partial\Lambda$
- $E = \{xy : x, y \in \Lambda, x \sim y\} \cup \{xy : x \in \Lambda \ y \in \partial\Lambda, x \sim y\}$

$E = \{\text{black edges}\} \cup \{\text{red edges}\}$.

Configurations.

$$\Omega^+ = \{\sigma \in \{-1,1\}^{\bar\Lambda} : \forall x \in \partial\Lambda \ \sigma_x = +1.\}$$



2 ISING MEASURE WITH + B.C.

Let $\beta \geq 0$ "inverse temperature".

Energy of $\sigma \in \Omega_+$: $H^+(\sigma) := -\beta \sum_{xy \in E} J_{xy} \sigma_x \sigma_y$

Ising measure in Λ, with + boundary conditions, at inverse-temperature β, with interactions J:

$$\forall \sigma \in \Omega^+ \quad \mu^+[\sigma] = \frac{1}{Z^+} \cdot e^{-H^+(\sigma)}$$

where $Z^+ = \sum_{\sigma \in \Omega^+} e^{-H^+(\sigma)}$ partition function.

Rks: • β represents the intensity of the interactions between the spins

• J_{xy} : interaction between σ_x and σ_y. ($J_{xy}=0$: no interaction.)

Particular cases.

- nearest neighbour (n.n.) interactions : $J_{xy} = \mathbb{1}_{x\sim y}$.

- free measure $J_{xy} = \begin{cases} 0 & \text{if } x \in \partial\Lambda \text{ or } y \in \partial\Lambda \\ \mathbb{1}_{x\sim y} & \text{if } x, y \in \Lambda \end{cases}$

Notation: for $f : \Omega^+ \to \mathbb{R}$ random variable, we write.

. $\langle f \rangle^+ = \dfrac{1}{Z^+} \sum_{\sigma \in \Omega^+} f(\sigma)\, e^{-H^+(\sigma)}$. "expectation w.r.t. μ^+"

. $Z^+[f] = \sum_{\sigma\in\Omega} f(\sigma) e^{-H^+(\sigma)}$.

Depending on the context, we may add the dependence on Λ, J, β and write $\mu^+_\beta, \mu^+_J, \mu^+_\Lambda, \mu^+_{\Lambda,\beta}, \Omega^+_\Lambda, Z^+_\Lambda, Z^+_\beta, \dots$

3 MULTI-POINT SPIN FUNCTIONS

Def: For $A \subset \Lambda$, define $\boxed{\sigma_A = \prod_{x\in A} \sigma_x}$.
(identified with the random variable $\sigma \mapsto \prod_{x\in A} \sigma_x$.)

Prop: $\boxed{(\sigma_A)_{A\subset\Lambda} \text{ forms a basis of } \mathbb{R}^{\Omega_+}.}$

Proof: Let $\mathbf{E}$ be the expectation on Ω_+ w.r.t the uniform measure. Let $A, B \subset \Lambda$.

$$\mathbf{E}[\sigma_A \sigma_B] = \mathbf{E}[\sigma_{\underset{\substack{\uparrow\\ \text{symmetric}\\ \text{difference}}}{A\Delta B}}] \underset{\substack{\uparrow\\ \text{indep.}}}{=} \prod_{i\in A\Delta B} \underbrace{\mathbf{E}[\sigma_i]}_{=0} = \begin{cases} 0 & \text{if } A \neq B \\ 1 & \text{if } A = B \end{cases}.$$

$(\sigma_A)_{A \subset \Lambda}$ is orthonormal for the inner product $(f;g) = \mathbb{E}[fg]$, it has $2^{|\Lambda|} = \dim(\mathbb{R}^{\Omega^+})$ elements, hence it is a basis. ∎

Any random variable f can be written as a linear combination $f = \sum_{A \subset \Lambda} f_A \cdot \sigma_A$, $f_A \in \mathbb{R}$.

→ the measure μ^+ is characterized by $(\langle \sigma_A \rangle^+)_{A \subset \Lambda}$.

CHAPTER 2 :

HIGH - TEMPERATURE EXPANSIONS.

$\Lambda \Subset \mathbb{Z}^d$, $\beta \geq 0$, J coupling constants ($J_{xy} \geq 0$)

General idea behind geometric representations:

goal: rewrite $\langle \sigma_A \rangle^+ = \dfrac{\sum_{\sigma \in \Omega^+} \sigma_A e^{-H^+(\sigma)}}{\sum_{\sigma \in \Omega^+} e^{-H^+(\sigma)}}$ as a sum

over different combinatorial objects. More precisely we consider a set of objects $\mathcal{X}$ and we want to write

$$\langle \sigma_A \rangle^+ = \frac{\sum_{x \in \mathcal{X}_A} f(x)\, w(x)}{\sum_{x \in \mathcal{X}_\emptyset} w(x)}$$

where $\mathcal{X}_\emptyset, \mathcal{X}_A \subset \mathcal{X}$, $f: \mathcal{X} \to \mathbb{R}$, $w: \mathcal{X} \to \mathbb{R}$ "weight function"

In the case $\mathcal{X}_\emptyset = \mathcal{X}_A$, this also has a probabilistic interpretation.

This is particularly powerful when the objects in $\mathcal{X}$ have "nice" combinatorial properties.

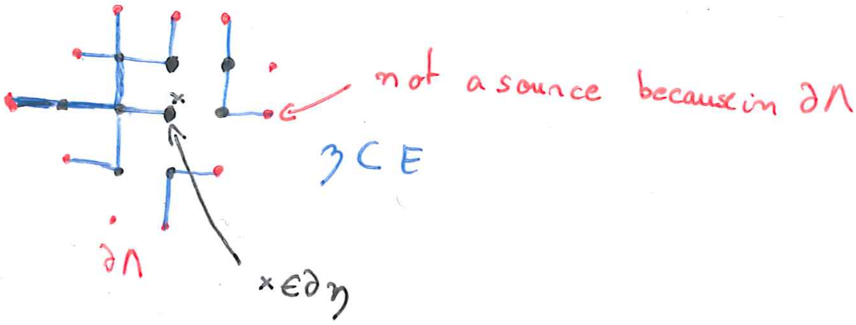
Notice that $\langle \sigma_A \rangle^+ = \dfrac{Z^+[\sigma_A]}{Z^+[1]}$. Hence, we only need

to rewrite $Z[\sigma_A] = \sum_{\sigma \in \Omega^+} \sigma_A e^{-H^+(\sigma)} = \sum_{x \in \mathcal{X}_A} f(x)\, w(x)$.

two techniques:
- write $\sigma_A e^{-H^+(\sigma)}$ as $\sum_{x \in \mathcal{X}} F(x, \sigma)$ and permut the sums.
- use a change of variable $x = \varphi(\sigma)$

1 SUBGRAPHS AND SOURCES

$\gamma \subset E$ "subgraph of $(\bar{\Lambda}, E)$"



write $\partial\gamma = \{ x \in \Lambda : \underbrace{\sum_{y \in \bar{\Lambda}} \mathbb{1}_{xy \in \gamma}}_{\#\text{ edges of } \gamma \text{ adjacent to } x} \text{ is odd} \}$ "sources of γ"

⚠ the sources are only the element $\underline{\underline{\text{of } \Lambda}}$ with odd degree

Rk: a graph $\gamma \subset E$ with $\partial\gamma$ is a graph where all the vertices in Λ have even degree.

2 HIGH TEMPERATURE EXPANSION

Thm: Let $A \subset \Lambda$. We have

$$\boxed{\langle \sigma_A \rangle^+ = \frac{\sum_{\gamma \subset E : \partial\gamma = A} w(\gamma)}{\sum_{\gamma \subset E : \partial\gamma = \emptyset} w(\gamma)}} \quad \text{where } w(\gamma) = \prod_{xy \in \gamma} \tanh(\beta J_{xy})$$

Rk: if $J_{xy} = \mathbb{1}_{x \sim y}$, we have $w(\gamma) = [\tanh(\beta)]^{|\gamma|}$

Lemma:

> Let I be a finite set, $(a_i)_{i\in I} \in \mathbb{R}^I$. Then,
> $$\prod_{i\in I}(1+a_i) = \sum_{\gamma\subset I}\Big(\prod_{i\in\gamma} a_i\Big).$$

Proof of the theorem.

We rely on the following elementary inequality.

$$\forall \varepsilon\in\{\pm1\}\ \forall\lambda\in\mathbb{R}\quad e^{\lambda\varepsilon} = \cosh(\lambda)+\varepsilon\sinh(\lambda)$$
$$= \cosh(\lambda)\big(1+\varepsilon\tanh(\lambda)\big) \qquad (*)$$

Let $\mathbf{E}$ be the expectation w.r.t. the uniform measure on Ω^+.

$$\frac{1}{|\Omega^+|}Z^+ = \frac{1}{|\Omega^+|}\sum_{\sigma\in\Omega^+} e^{-H^+(\sigma)}$$

$$= \mathbf{E}\Big[\prod_{xy\in E} e^{\beta J_{xy}\sigma_x\sigma_y}\Big]$$

$$\overset{(*)}{=} \mathbf{E}\Big[\underbrace{\prod_{xy\in E}\cosh(\beta J_{xy})}_{=:C}\prod_{xy\in E}\big(1+\sigma_x\sigma_y\tanh(\beta J_{xy})\big)\Big]$$

$$\overset{\text{Lemma}}{=} C\,\mathbf{E}\Big[\sum_{\gamma\subset E}\prod_{xy\in\gamma}\sigma_x\sigma_y\tanh(\beta J_{xy})\Big]$$

$$= C\sum_{\gamma\subset E}\prod_{xy\in\gamma}\tanh(\beta J_{xy})\,\underbrace{\mathbf{E}\Big[\prod_{x\in\Lambda}\sigma_x^{\sum_{y\in\Lambda}\mathbb{1}_{xy\in\gamma}}\Big]}_{(1)}$$

By independence (1) $= \prod_{x\in\Lambda} \underbrace{\mathbb{E}\left[\sigma_x^{\sum_{y\in\Lambda} \mathbb{1}_{xy\in\gamma}}\right]}_{=0 \text{ if } x\in\partial\gamma}$

$= \mathbb{1}_{\partial\gamma=\emptyset}$.

Finally $Z^+ = |\Omega^+| \times C \times \sum_{\gamma:\partial\gamma=\emptyset} w(\gamma)$.

Equivalently $Z^+[\sigma_A] = |\Omega^+| \times C \times \sum_{\gamma:\partial\gamma=A} w(\gamma)$,

which concludes the proof since $\langle\sigma_A\rangle^+ = \dfrac{Z^+[\sigma_A]}{Z^+}$. ∎

3. APPLICATION IN DIMENSION 1.

<u>Proposition</u>: Let $d=1$, $\Lambda_n = \{-n,\dots,n\}$, $J = (\mathbb{1}_{x\sim y})$

$$\boxed{\langle\sigma_0\rangle^+_{\Lambda_n} = 2\,\frac{\tanh(\beta)^{n+1}}{1+\tanh(\beta)^{2n+2}} \leq 2\tanh(\beta)^{n+1}}$$

<u>Proof</u>: subgraphs with $\partial\gamma=\emptyset$:  , 

$\gamma = E$ $\qquad$ $\gamma=\emptyset$

hence $\sum_{\gamma:\partial\gamma=\emptyset} \tanh(\beta)^{|\gamma|} = \tanh(\beta)^{2n+2} + 1$.

Subgraphs with $\partial\gamma=\{0\}$:  , 

$\Leftrightarrow \sum_{\gamma:\partial\gamma=\{0\}} \tanh(\beta)^{|\gamma|} = 2\times\tanh(\beta)^{n+1}$ ∎

4. GKS INEQUALITIES.

(GKS : Griffiths, Kelly, Sherman)

Theorem

Let $A, B \subset \Lambda$. Then

(i) $\langle \sigma_A \rangle^+ \geq 0$ [1st Griffiths]

(ii) $\langle \sigma_A \sigma_B \rangle^+ \geq \langle \sigma_A \rangle^+ \langle \sigma_B \rangle^+$ [2nd Griffiths.]

Proof: (i) $\langle \sigma_A \rangle^+ = \dfrac{\sum_{\eta : \partial\eta = A} w(\eta)}{\sum_{\eta : \partial\eta = 0} w(\eta)} \geq 0$ because $\forall \eta$ $w(\eta) \geq 0$.

(ii) $Z^+[\sigma_A \sigma_B]\, Z^+[1] - Z^+[\sigma_A]\, Z^+[\sigma_B]$

$$= \sum_{\sigma, \sigma' \in \Omega^+} (\sigma_A \sigma_B - \sigma_A \sigma'_B)\, e^{-H^+(\sigma) - H^+(\sigma')}$$

$$= \sum_{\sigma, \sigma' \in \Omega^+} \sigma_A \sigma_B (1 - \sigma_B \sigma'_B)\, e^{\beta \sum_{xy \in E} J_{xy} \sigma_x \sigma_y (1 - \sigma_x \sigma_y \sigma'_x \sigma'_y)}$$

$$= \sum_{\omega \in \Omega^+} (1 - \omega_B) \underbrace{\sum_{\sigma \in \Omega^+} \sigma_A \sigma_B\, e^{\beta \sum_{xy} (1 - \omega_x \omega_y) J_{xy} \sigma_x \sigma_y}}_{= \cancel{\langle \sigma_A \sigma_B \rangle^+_{J^\omega}}\ Z^+_{J^\omega}[\sigma_A \sigma_B]^\omega}$$

where $J^\omega_{xy} = (1 - \omega_x \omega_y) J_{xy} \geq 0$

≥ 0

by 1st Griffiths

5 MONOTONICITY PROPERTIES

Write $J \leq J'$ if $\forall xy \quad J_{xy} \leq J'_{xy}$

Prop. [Monotonicity in J and β]

> Let $\beta \leq \beta'$ and $J \leq J'$, then
>
> $$\forall A \subset \Lambda \quad \langle \sigma_A \rangle^+_{J,\beta} \leq \langle \sigma_A \rangle^+_{J',\beta'} .$$

"stronger interactions implies more pluses"

Proof:
$$Z^+_{J',\beta'}[\sigma_A] = \sum_{\sigma \in \Omega^+} \sigma_A \, e^{\beta' \sum_{xy} J'_{xy} \sigma_x \sigma_y}$$

$$= \sum_{\sigma \in \Omega_+} \sigma_A \, g(\sigma) \, e^{\beta \sum_{xy} J_{xy} \sigma_x \sigma_y} \qquad = Z^+_{J,\beta}[\sigma_A g]$$

where $g(\sigma) = \exp\left(\sum_{xy} (\beta' J'_{xy} - \beta J_{xy}) \sigma_x \sigma_y \right)$

Hence $\langle \sigma_A \rangle^+_{J',\beta'} = \dfrac{Z^+_{J,\beta}[\sigma_A g]}{Z^+_{J,\beta}[g]} = \dfrac{\langle \sigma_A g \rangle^+_{J,\beta}}{\langle g \rangle^+_{J,\beta}}$

Observe that $g(\sigma) = \sum_k \frac{1}{k!} \left(\sum_{xy} (\beta' J'_{xy} - \beta J_{xy}) \sigma_x \sigma_y \right)^k$

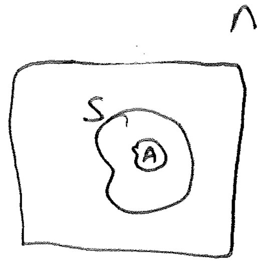
$$= \sum_{S \subset \Lambda} \alpha_S \sigma_S \qquad \text{where } \alpha_S \geq 0 .$$

Hence, by GKS inequality and linearity

$$\langle \sigma_A g \rangle_{J,\beta} \geq \langle \sigma_A \rangle_{J,\beta} \langle g \rangle_{J,\beta} ,$$

which concludes $\langle \sigma_A \rangle^+_{J',\beta'} \geq \langle \sigma_A \rangle^+_{J,\beta}$. ∎

Prop [Monotonicity in Λ]



> Let $A \subset S \subset \Lambda$. Then
>
> $$\langle \sigma_A \rangle_S^+ \geq \langle \sigma_A \rangle_\Lambda^+$$

" the spins are more aligned on + if the b.c. are closer "

Lemma:

> Let $A \subset S \subset \Lambda$ Then. $\langle \sigma_A \rangle_S^+ = \langle \sigma_A \mid \forall x \in \Lambda \setminus S \ \sigma_x = +1 \rangle_\Lambda^+$

Rk: this is a particular case of a general property of the Ising measures.

Proof of the Lemma:

Let $F = \{ xy \in E_\Lambda : x \notin S \ y \notin S \}$

Observe that $\forall \sigma$ s.t. $\forall x \in \Lambda \setminus S \quad \sigma_x = +1$

$$H_\Lambda(\sigma) = H_S(\sigma) - \beta |F|$$

Hence $\langle \sigma_A \mid \forall x \in \Lambda \setminus S \ \sigma_x = +1 \rangle_\Lambda^+ = \dfrac{\sum_{\sigma : \forall x \notin S \ \sigma_x = +1} \sigma_A e^{-H_S(\sigma) + \beta |F|}}{\sum_{\sigma : \forall x \notin S \ \sigma_x = +1} e^{-H_S(\sigma) + \beta |F|}}$

$$= \langle \sigma_A \rangle_S^+ \qquad \blacksquare$$

Proof of the proposition:

Observe that $\mathbb{1}_{\forall x \in \Lambda \setminus S\ \sigma_x = +1} = \prod_{x \in \Lambda \setminus S} \left(\frac{1+\sigma_x}{2} \right) = \frac{1}{2^{|\Lambda \setminus S|}} \sum_{T \subset \Lambda \setminus S} \sigma_T$.

By the Lemma

$$\langle \sigma_A \rangle_S^+ = \frac{\langle \sigma_A \sum_{T \subset \Lambda \setminus S} \sigma_T \rangle_\Lambda^+}{\langle \sum_{T \subset \Lambda \setminus S} \sigma_T \rangle_\Lambda^+}$$

$$\overset{GKS}{\geqslant} \frac{\sum_{T \subset \Lambda \setminus S} \langle \sigma_A \rangle_\Lambda^+ \langle \sigma_T \rangle_\Lambda^+}{\sum_{T \subset \Lambda \setminus S} \langle \sigma_T \rangle_\Lambda^+} \geqslant \langle \sigma_A \rangle_\Lambda^+ \qquad \blacksquare$$

CHAP. 4 STOCHASTIC DOMINATION

Let Ω be a Polish space, equipped with a partial ordering $\leq$. and its Borel σ-algebra.

1 Definition and first examples.

Def: Let μ, ν be two probability measures on Ω.
We say that μ is stochastically dominated by ν (written $\mu \ll \nu$) if for every $f : \Omega \longrightarrow \mathbb{R}$ increasing measurable bounded

$$\int f \, d\mu \leq \int f \, d\nu .$$

Exple 1

$\Omega = \mathbb{R}$. For $x > 0$ consider μ_x the law of a uniform random variable on $[0, x]$ (ie $d\mu_x = \mathbb{1}_{[0,x]} \times \frac{1}{x} \, dt$)

Then $0 < x \leq y \Rightarrow \mu_x \ll \mu_y$

Proof: Let X be a uniform r.v. on $[0, x]$

Then $Y = \frac{y}{x} \cdot X$ is a uniform r.v on $[0, y]$

since almost surely $X \leq Y$

we have for every $f \nearrow$ measurable bounded

$$f(X) \leq f(Y) \text{ a.s.}$$

Therefore, by taking the expectation, we obtain

$$\underbrace{E[f(X)]}_{\displaystyle \overset{\shortparallel}{\int f \, d\mu_x}} \leq \underbrace{E[f(Y)]}_{\displaystyle \overset{\shortparallel}{\int f \, d\mu_y}}$$

Exple 2:

$\Omega = \{0,1\}$. For $p \in [0,1]$, let $\mu_p = \text{Bernoulli}(p)$.

Then $0 \leq p \leq q \leq 1 \Rightarrow \mu_p \ll \mu_q$

Proof:

Let U be a uniform random variable in $[0,1]$.

Define $X = \begin{cases} 1 & \text{if } U \leq p \\ 0 & \text{if } U > p \end{cases}$ and $Y = \begin{cases} 1 & \text{if } U \leq q \\ 0 & \text{if } U > q \end{cases}$

Then $X \sim \mu_p$ and $Y \sim \mu_q$

$$p < q \Rightarrow X \leq Y \text{ a.s.} \Rightarrow \forall f : \Omega \to \mathbb{R} \uparrow \quad f(X) \leq f(Y)$$

$$\Rightarrow \underbrace{\mathbb{E}[f(X)]}_{\int f \, d\mu_p} \leq \underbrace{\mathbb{E}[f(Y)]}_{\int f \, d\mu_q}. \qquad \square$$

In the two examples above, we relied on a coupling method to show the stochastic domination.

Def: Let (E,μ), (F,ν) be two probability space. We call coupling of μ and ν a probability measure P on the product space $E \times F$ such that

- its first marginal is μ ($P[A \times F] = \mu(A)$ $\forall A$ measurable)
- its second marginal is ν ($P(E \times B) = \nu(B)$ $\forall B$ measurable)

In the two examples above we prove $\mu \ll \nu$ by constructing a coupling P (on $\Omega \times \Omega$) of μ and ν such that

$$P\left(\{(\omega,\eta) \in \Omega \times \Omega : \omega \leq \eta\}\right) = 1$$

(P is the law of the pair (X,Y))

This easily implies the desired stochastic domination. For all the applications in this course, we will always prove stochastic domination by constructing a coupling of the two measures. Actually, Strassen's theorem states that the reciprocal is also true if Ω is a Polish space.

Theorem: Assume that Ω is Polish. Let μ, ν be two probability measures on Ω. The following are equivalent:

(i) $\mu \ll \nu$

(ii) there exists a coupling P of μ and ν such that

$$P\left(\{(\omega,\eta) \in \Omega \times \Omega : \omega \leq \eta\}\right) = 1$$

(iii) there exist two random variables $X \sim \mu$ and $Y \sim \nu$ on the same probability space and such that

$$X \leq Y \quad \text{a.s.}$$

Proof:

(ii) $\Rightarrow$ (iii) consider a random variable (X,Y) on $\Omega \times \Omega$ with law P.

(iii) $\Rightarrow$ (i) if $X \leq Y$ a.s.
then for every $f : \Omega \to \mathbb{R}$ increasing measurable bounded, we have

$$f(X) \leq f(Y) \quad \text{a.s.}$$

taking the expectation gives $\int f d\mu \leq \int f d\nu$.

(i) $\Rightarrow$ (ii) see e.g. Lindvall'99 (e.c.p) or Werner (percolation et modèle d'Ising p. 98) for the case Ω finite.

2. STOCHASTIC DOMINATION ON PRODUCT SPACES.

In this section, we fix a finite set S, and consider $\Omega = \{0,1\}^S$ equipped with the product ordering $\leq$.

$$\eta \leq \psi \iff \forall i \in S \ \eta_i \leq \psi_i .$$

Exercice:

For $p \in [0,1)$ let $\mu_p = \text{Bernoulli}(p)^{\otimes S}$ be the law of $X = (X_s)_{s\in S}$ where X_s are iid Bernoulli variables with parameter p.
Prove that $p \leq q \iff \mu_p \ll \mu_q$...

For non-product measures (which correspond to random variables $X = (X_s)_{s \in S}$ with dependencies, X_s may depend on X_t (for $s \neq t$)), it is a priori not obvious to prove stochastic domination. We present here the Holley criterion which is a powerful tool in order to prove stochastic dominations on Ω.

Thm: (Holley criterion).

Let μ, ν be two positive measures on Ω (i.e. $\mu(\eta), \nu(\eta) > 0$ for every $\eta \in \Omega$). Assume that for every $\eta \leq \psi$

$$\frac{\mu[\eta^e]}{\mu[\eta_e]} \leq \frac{\nu[\psi^e]}{\nu[\psi_e]}$$

Then $\mu \ll \nu$.

Préliminary:

The proof of the theorem is based on a Markov chain method. In order to construct a suitable coupling for μ and ν, we will couple two Markov Chains $X = (X_n)_n$ and $Y = (Y_n)_n$ with respective invariant measures μ and ν, s.t.

$X_n \leq Y_n$ for every n.

Before that, let use describe one Markov chain $(X_n)_n$ with invariant measure μ.

The chain starts from a fixed configuration $X_0 = \eta_0$.

Then for $n \geq 0$ X_{n+1} is constructed from X_n as follows.

Pick $S_n \in S$ uniformly at random.

Define . for $s \neq S_n$ $X_{n+1}(s) = X_n(s)$

. for $s = S_n$

$$X_{n+1}(s) = \begin{cases} 1 & \text{with prob. } \mu[\omega(s)=1 \mid \forall t \neq s \ \omega(t) = X_n(t)] \\ 0 & \text{otherwise} \end{cases}$$

One can check that (X_n) is an irreducible aperiodic Markov Chain on Ω with invariant measure μ.

In particular $E[f(X_n)] \xrightarrow[n \to \infty]{} \int f \, d\mu$

Proof of Thm:

Let $\eta \leq \psi$ be two configurations in Ω, let $s \in S$.

$$\mu\left[\omega(s)=1 \mid \forall t \in S\setminus\{s\}\ \omega(t)=\eta(t)\right]$$

$$= \frac{\mu(\eta^s)}{\mu(\eta_s)+\mu(\eta^s)} = \frac{1}{1+\mu(\eta_s)/\mu(\eta^s)}$$

$$\leq \frac{1}{1+\nu(\psi_s)/\nu(\psi^s)} = \nu\left[\omega(s)=1 \mid \forall t \in S\setminus\{s\}\ \omega(t)=\psi(t)\right] \qquad (*)$$

Let $S_1, \dots, S_n, \dots$ be an iid sequence of uniform random variables in S : for every $s \in S$ $\mathbb{P}[S_i = s] = \frac{1}{|S|}$.

Let $U_1, \dots, U_n, \dots$ be an iid sequence of uniform random variables in $[0,1]$.

We construct a Markov Chain (X_n, Y_n) on $\Omega \times \Omega$ as follows. Fix a configuration $\eta_0 \in \Omega$ and set

$$(X_0, Y_0) = (\eta_0, \eta_0).$$

For $n \geq 0$, define (X_{n+1}, Y_{n+1}) as follows:

For $s \neq S_{n+1}$, set $X_{n+1}(s) = X_n(s)$ and $Y_{n+1}(s) = Y_n(s)$.

For $s = S_{n+1}$ set

$$X_{n+1}(s) = \begin{cases} 1 & \text{if } U_{n+1} \leq \mu\left[\omega(s)=1 \mid \forall t\neq s\ \omega(t)=X_n(s)\right] \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_{n+1}(s) = \begin{cases} 1 & \text{if } U_{n+1} \leq \nu\left[\omega(s)=1 \mid \forall t\neq s\ \omega(t)=Y_n(s)\right] \\ 0 & \text{otherwise} \end{cases}$$

Using $(*)$, we have by induction that

$$X_n \leq Y_n \text{ for every } n \geq 1.$$

Therefore, for every $f : \Omega \to \mathbb{R}$ increasing

$$\mathbb{E}[f(X_n)] \leq \mathbb{E}[f(Y_n)]$$

One can check that

- X_n is an irreducible Markov Chain on Ω (because μ is positive, it is possible to move from one configuration to an other by replacing bits by bits all the places where the configurations differ)

- μ is an invariant measure (because it is reversible : for every η and every $s \in S$.

$$\mu(\eta^s) \cdot \mathbb{P}[X_{n+1} = \eta_s \mid X_n = \eta^s]$$

$$= \mu(\eta^s) \cdot \frac{1}{|S|} \cdot \frac{\mu(\eta_s)}{\mu(\eta_s) + \mu(\eta^s)}$$

$$= \mathbb{P}[X_{n+1} = \eta^s \mid X_n = \eta_s) \ \mu(\eta_s) \ )$$

Hence $\mathbb{E}[f(X_n)] \xrightarrow[n \to \infty]{} \int f \, d\mu$.

And equivalently $\mathbb{E}[f(Y_n)] \longrightarrow \int f \, d\nu$ $\blacksquare$

Rk: In the proof above, it is actually possible to construct a coupling P of μ and ν s.t. $P[\{(\omega,\eta) : \omega \leq \eta\}] = 1$ by considering an invariant measure of the Markov chain (X_n, Y_n) on $\Omega \times \Omega$. (Exercice).

Exercice:

Let $\Lambda \subset\subset \mathbb{Z}^d$, $\xi, \psi \in \{0,1\}^{\mathbb{Z}^d}$ b.c.

Prove using Holley-Criterion that

$$\xi \leq \psi \Rightarrow \mu^{\xi}_{\Lambda,\beta,h} \ll \mu^{\psi}_{\Lambda,\beta,h}$$

$$h \leq h' \Rightarrow \mu^{\xi}_{\Lambda,\beta,h} \ll \mu^{\xi}_{\Lambda,\beta,h'}$$

(Above we see the Ising measures as measures on $\Omega = \{0,1\}^{\Lambda}$, via the identification $\Omega \overset{bij}{\simeq} \Omega^{\xi}_{\Lambda} \overset{bij}{\simeq} \Omega^{\psi}_{\Lambda}$)

FKG INEQUALITY

In this section, we give a criterion, based on the Holley criterion, that allows one to prove FKG inequality for dependent measures.

Thm: Let μ be a positive measure on $\Omega = \{0,1\}^S$ s.t.

$$\forall s \in S \quad \forall \eta \leq \psi \quad \frac{\mu(\eta^s)}{\mu(\eta_s)} \leq \frac{\mu(\psi^s)}{\mu(\psi_s)}$$

Then μ satisfies the FKG-inequality:

$$\forall f, g : \Omega \to \mathbb{R} \text{ increasing} \quad \int fg \, d\mu \geq \int f d\mu \cdot \int g \, d\mu$$

Proof: Without loss of generality we can assume that $f(\omega) > 0 \quad \forall \omega \in \Omega$ (if not, consider $f + c$, where c is a large constant).

Consider the positive probability measure ν defined by

$$\forall \psi \in \Omega \quad \nu(\psi) := \frac{1}{\int f \, d\mu} \cdot f(\psi) \cdot \mu(\psi)$$

Since f is increasing we have, for every $\psi \geq \eta$

$$\frac{\nu(\psi^e)}{\nu(\psi_e)} = \underbrace{\frac{f(\psi^e)}{f(\psi_e)}}_{\geq 1} \cdot \frac{\mu(\psi^e)}{\mu(\psi_e)} \geq \frac{\mu(\eta^e)}{\mu(\eta_e)}$$

Therefore, by Holley criterion, $\mu \ll \nu$ and

$$\int g \, d\mu \leq \int g \, d\nu = \frac{1}{\int f \, d\mu} \int f g \, d\mu$$

which concludes the proof ■

Application: proof of the FKG-inequality for the Ising measure.

Let $\Lambda \subset\subset \mathbb{Z}^d$ and $\Omega_\Lambda = \{0,1\}^\Lambda$. $\mu = \mu^{\emptyset}_{\Lambda,\beta,h}$.

For $\sigma \in \Omega_\Lambda$ we have for $i \in \Lambda$

$$H^{\emptyset}_{\Lambda,\beta,h}(\sigma^i) - H^{\emptyset}_{\Lambda,\beta,h}(\sigma_i) = -\beta \sum_{j:\{i,j\} \in E_\Lambda} (\sigma^i - \sigma_i) \cdot \sigma_j - h(\sigma^i - \sigma_i)$$

$$= -2\beta \sum_{j:\{i,j\} \in E_\Lambda} \sigma_j - 2h$$

CHAPTER 5 : FINITE VOLUME ISING MEASURES WITH CONFIGURATIONAL BOUNDARY CONDITIONS.

$\Lambda \subset \mathbb{Z}^d$ finite

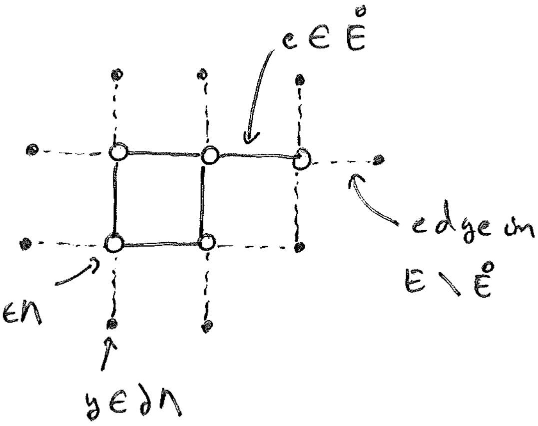
$\bar{\Lambda} = \Lambda \cup \partial\Lambda$

edges $E = \{xy,\ x \in \bar{\Lambda}\ y \in \bar{\Lambda},\ x \sim y\}$

$\mathring{E} = \{xy \in E \quad x \in \Lambda\ y \in \Lambda\}$

$J_{xy} = \mathbb{1}_{x \sim y}$ n.n interactions

$\beta \geq 0 \qquad h \in \mathbb{R}$.



1 DEFINITIONS.

Def: : boundary condition (b.c.) for Λ : $\omega \in \{+1,-1\}^{\partial\Lambda}$

spin configuration $\sigma \in \Omega := \{+1,-1\}^{\Lambda}$

Def: The Ising measure in Λ with b.c. ω, inverse temperature β, external field h is defined by

$$\forall \sigma \in \Omega \qquad \mu^{\omega}[\sigma] = \frac{1}{Z^{\omega}} e^{-H^{\omega}(\sigma)}$$

where . $\boxed{H^{\omega}(\sigma) = -\beta \sum_{xy \in \mathring{E}} \sigma_x \sigma_y - \beta \sum_{\substack{xy \in E \\ y \in \partial\Lambda}} \sigma_x \omega_y - h \sum_{x \in \Lambda} \sigma_x}$

$$\cdot\ Z^{\omega} = \sum_{\sigma \in \Omega} e^{-H^{\omega}(\sigma)}$$

Not: $\mu^{\omega} = \mu^{\omega}_{\Lambda,\beta,h} = \mu^{\omega}_{\Lambda} = \mu^{\omega}_{\beta} \ldots$

$\langle f \rangle^{\omega} = \int_{\Omega} f \, d\mu^{\omega} = \sum_{\sigma \in \Omega} f(\sigma)\, \mu^{\omega}(\sigma)$.

2 COMPARISON BETWEEN B.C.

<u>Prop</u>: If $\omega \leq \omega'$, $h \leq h'$, then $\mu_h^\omega \ll \mu_h^{\omega'}$

<u>Rk</u>: In particular $\forall \omega$ b.c. $\mu^- \ll \mu^\omega \ll \mu^+$

<u>Application</u>: $\langle \sigma_0 \rangle_h^\omega \leq \langle \sigma_0 \rangle_h^{\omega'}$ if $\omega \leq \omega'$ and $h \leq h'$.

<u>Proof</u>: Write $\sigma^{(x)}$ (resp. $\sigma_{(x)}$) for the configuration equal to σ at every vertex except possibly at x where it takes the value $+1$ (resp. -1).

Let $\sigma' \geq \sigma$

$$H_h^\omega(\sigma_{(x)}) - H_h^\omega(\sigma^{(x)}) = 2\beta\Big(\sum_{\substack{y \sim x \\ y \in \Lambda}} \sigma_y + \sum_{\substack{y \sim x \\ y \in \partial\Lambda}} \omega_y\Big) + 2h$$

$$\leq 2\beta\Big(\sum_{\substack{y \sim x \\ y \in \Lambda}} \sigma'_y + \sum_{\substack{y \sim x \\ y \in \partial\Lambda}} \omega'_y\Big) + 2h'$$

$$= H_{h'}^{\omega'}(\sigma'_{(x)}) - H_{h'}^{\omega'}(\sigma'^{(x)})$$

Hence $$\frac{\mu_h^\omega(\sigma^{(x)})}{\mu_h^\omega(\sigma_{(x)})} \leq \frac{\mu_{h'}^{\omega'}(\sigma'^{(x)})}{\mu_{h'}^{\omega'}(\sigma'_{(x)})}$$

Holley criterion concludes the proof. ∎

3. DOMAIN MARKOV PROPERTY

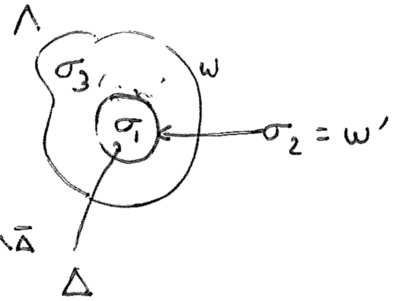
Prop: Let $\Delta \subset \Lambda$, ω' b.c. for Δ compatible with ω

(ie $\forall x \ \partial\Lambda \cap \partial\Delta \quad \omega'(x) = \omega(x)$)

Then

$$\forall \eta \in \Omega_\Delta \,, \quad \mu^{\omega}_{\Lambda}\left[\sigma_{|\Delta} = \eta \,\middle|\, \forall x \in \partial\Delta \cap \Lambda \ \ \sigma_x = \omega'_x\right] = \mu^{\omega'}_{\Delta}[\eta]$$

Proof: For simplicity we assume $\partial\Delta \subset \Lambda$



We decompose each configuration

$\sigma \in \Omega_\Lambda$ onto $\sigma_1 \in \Omega_\Delta$ $\sigma_2 \in \Omega_{\partial\Delta}$ $\sigma_3 \in \Omega_{\Lambda \setminus \bar{\Delta}}$

This way, we have $\forall \sigma : \ \sigma_2 = \omega'$

$$H^{\omega}_{\Lambda}(\sigma) = H^{\omega'}_{\Delta}(\sigma_1) + H^{\omega,\omega'}_{\Lambda\setminus\bar{\Delta}}(\sigma_3) \underbrace{- \beta \sum_{xy \in E_{\partial\Delta}} \omega'_x \omega'_y - h \sum_{x \in \partial\Lambda} \omega'_x}_{=: \, C(\omega')}$$

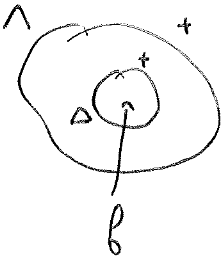
Therefore
$$Z^{\omega}_{\Lambda}[\sigma_1 = \eta, \sigma_2 = \omega'] = Z^{\omega,\omega'}_{\Lambda\setminus\bar{\Delta}} \times e^{-C(\omega')} \times e^{-H^{\omega'}_{\Delta}(\eta)}$$

and
$$Z^{\omega}_{\Lambda}[\sigma_2 = \omega'] = Z^{\omega,\omega'}_{\Lambda\setminus\bar{\Delta}} \cdot e^{-C(\omega')} \times Z^{\omega'}_{\Delta}$$

Hence
$$\underbrace{\frac{Z^{\omega}_{\Lambda}[\sigma_1 = \eta, \sigma_2 = \omega']}{Z^{\omega}_{\Lambda}[\sigma_2 = \omega']}}_{\mu^{\omega}_{\Lambda}\left[\sigma_{|\Delta} = \eta \,\middle|\, \forall x \in \partial\Delta \ \sigma_x = \omega'_x\right]} = \mu^{\omega'}_{\Delta}[\eta]$$

Exercise: Let $\Delta \subset \Lambda$. Let f be an increasing function of $(\sigma_i)_{i \in \Delta}$. Then

$$\langle f \rangle_\Lambda^+ \leq \langle f \rangle_\Delta^+$$



4. FKG INEQUALITY.

Def: A function $f : \Omega_\Lambda \longrightarrow \mathbb{R}$ is said to be increasing if $\sigma \leq \sigma' \Rightarrow f(\sigma) \leq f(\sigma')$

Exples
- $\sigma \mapsto \sigma_0$ is increasing
- $\sigma \mapsto \sigma_A$ is NOT increasing if $|A| \geq 2$.
- $\forall x \in \mathbb{Z}^d$ $n_x : \sigma \longrightarrow \dfrac{\sigma_x + 1}{2}$ is increasing.
- $\forall A \subset \Lambda$ $\quad n_A := \underbrace{\prod_{x \in A} n_x}_{= \mathbb{1}_{\forall x \in A\ \sigma_x = +1}}$ is increasing.

Prop: $(n_A)_{A \subset \Lambda}$ is a basis of $\mathbb{R}^{\Omega_\Lambda} = \{ f : \Omega_\Lambda \longrightarrow \mathbb{R} \}$.

Proof:
- $\# \{ n_A \}_{A \subset \Lambda} = 2^{|\Lambda|} = \dim \mathbb{R}^{\Omega_\Lambda}$
- $\sigma_B = \prod_{x \in B} (2n_x - 1) \in \mathrm{Vect}(\{n_A\}_{A \subset \Lambda})$

hence $(n_A)_A$ is generating. ∎

Thm [FKG inequality]

$$\boxed{\begin{array}{l} \forall f, g : \Omega_\Lambda \longrightarrow \mathbb{R} \quad \text{increasing} . \\ \langle f g \rangle^\omega \geq \langle f \rangle^\omega \langle g \rangle^\omega \end{array}}$$

Proof: $\forall \sigma \leq \sigma'$, we have $\dfrac{\mu^\omega(\sigma^{(x)})}{\mu^\omega(\sigma_{(x)})} \leq \dfrac{\mu^\omega(\sigma'^{(x)})}{\mu^\omega(\sigma'_{(x)})}$ ■

Rk.: We say that $f : \Omega_\Lambda \to \mathbb{R}$ is decreasing if $-f$ is increasing.

$\forall f, g \searrow \qquad \langle f g \rangle^\omega \geq \langle f \rangle^\omega \langle g \rangle^\omega$

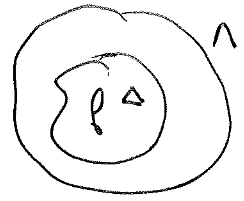
$\forall f \nearrow \ \forall g \searrow \quad \langle f g \rangle^\omega \leq \langle f \rangle^\omega \langle g \rangle^\omega$

Application: (Pushing B.C.)

Let $\Delta \subset \Lambda$, let f be an increasing function of $(\sigma_i)_{i \in \Delta}$

Then $\quad \langle f \rangle^+_\Delta \geq \langle f \rangle^+_\Lambda$

$\quad \langle f \rangle^-_\Delta \leq \langle f \rangle^-_\Lambda$



Proof: $\langle f \rangle^+_\Delta \overset{DMP}{=} \dfrac{\langle f \times n_{\partial\Delta} \rangle^+_\Lambda}{\langle n_{\partial\Delta} \rangle^+_\Lambda} \overset{FKG}{\geq} \langle f \rangle^+_\Lambda$

equivalently for $\langle \ \rangle^-$ replacing $n_{\partial\Delta}$ by the decreasing fct $\mathbb{1}_{\forall x \in \partial\Delta \ \sigma_x = -1}$. ■

Rk FKG : $\forall$ w B.C. $\langle n_A n_B \rangle^w \geq \langle n_A \rangle^w \langle n_B \rangle^w$.

GKS : $\langle \sigma_A \sigma_B \rangle^+ \geq \langle \sigma_A \rangle^+ \langle \sigma_B \rangle^+$.

$\langle \sigma_A \sigma_B \rangle^\emptyset \geq \langle \sigma_A \rangle^\emptyset \langle \sigma_B \rangle^\emptyset$

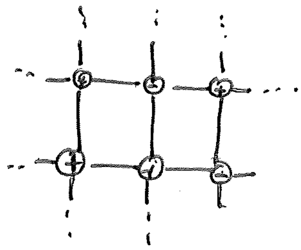
$\to$ not true for general B.C.

CHAPTER 6 :

ISING MODEL IN INFINITE VOLUME

$\Omega = \{+1,-1\}^{\mathbb{Z}^d}$ $\mathcal{F}$ product σ-algebra

$\beta \geq 0$ $h \in \mathbb{R}$ $J_{xy} = 1_{x\sim y}$ n.n. interaction



<u>Goal</u>: define μ Ising measure on $(\Omega, \mathcal{F})$.

For $\Lambda \subset\subset \mathbb{Z}^d$ $\rightsquigarrow$ $\mu^{\omega}_{\Lambda,\beta,h}$ in Ω_Λ

<u>idea 1</u> taking weak limits of μ^{w}_{Λ} as $\Lambda \uparrow \mathbb{Z}^d$

"this is the way we will define μ^+, μ^-"

<u>idea 2</u> via specification (Gibbs formalism).

Call μ an Ising measure on $\mathbb{Z}^d$ if its marginals in finite boxes (when we condition to the configuration outside the box) coincide with the finite volume Ising measures.

<u>1 PRELIMINARIES</u>

In order to construct infinite volume measures, we "need" an extension theorem from measure theory. Since we are working with a product space $\Omega = \{+1,-1\}^{\mathbb{Z}^d}$ we use Kolmogorov's extension theorem. (other approaches can be used, e.g. Riesz theorem, see [Velenik])

Not. $\mathcal{F}_\Lambda = \sigma\left((\sigma_i)_{i\in\Lambda} \right)$.

Rk: $\mathcal{F} = \sigma\left(\bigcup_{\Lambda\subset\subset \mathbb{Z}^d} \mathcal{F}_\Lambda \right)$

Thm: [Kolmogorov extension's theorem]

Consider a function $\mu : \bigcup_{\Lambda\subset\subset\mathbb{Z}^d} \mathcal{F}_\Lambda \longrightarrow \mathbb{R}_+$ s.t.

$\forall \Lambda \subset\subset \mathbb{Z}^d$ $\mu_{|\mathcal{F}_\Lambda}$ is a probability measure on $(\Omega, \mathcal{F}_\Lambda)$.

Then there exists a unique probability measure $\overline{\mu}$ on $\mathcal{F}$ that coincides with μ on every $\mathcal{F}_\Lambda$, $\Lambda\subset\subset\mathbb{Z}^d$.

Ref: The version above is taken from Villani's lecture notes (available at cedricvillani.org/for-mathematicians/lecture-notes [section III.6.5], in French)

Def: A function $f : \Omega \to \mathbb{R}$ is said to be local if there exists $\Lambda\subset\subset\mathbb{Z}^d$ s.t. f is $\mathcal{F}_\Lambda$-measurable.

Rk: The set of local functions is a vector space generated by $(n_A)_{A\subset\subset\mathbb{Z}^d}$.

Rk: If f is $\mathcal{F}_\Lambda$-measurable, then $f = f((\sigma_x)_{x\in\Lambda})$, and therefore f can be seen as a function $f : \Omega_\Lambda \to \mathbb{R}$

$\hookrightarrow \langle f \rangle^\omega_\Lambda$ is well defined.

2. THE INFINITE VOLUME MEASURES μ^+ AND μ^-.

Not: $\Lambda_k \uparrow \mathbb{Z}^d$ if $\Lambda_k \subset \Lambda_{k+1}$ and $\mathbb{Z}^d = \bigcup_{k \geq 1} \Lambda_k$

Thm: "[$\beta \geq 0$, $h \in \mathbb{R}$] There exist two probability measures μ^- and μ^+ on $(\Omega, \mathcal{F})$ characterized by $\forall f$ local function $\forall \Lambda_n \uparrow \mathbb{Z}^d$

(i) $\langle f \rangle^+ = \lim_{k \to \infty} \langle f \rangle^+_{\Lambda_k}$

(ii) $\langle f \rangle^- = \lim_{k \to \infty} \langle f \rangle^-_{\Lambda_k}$

where $\langle f \rangle^\omega = \int_\Omega f \, d\mu^\omega$ for $\omega \in \{+,-\}$

Rk: ① If f local, then $\langle f \rangle^+_{\Lambda_k}$ is well defined for k large.

② In (i) and (ii) the limit does not depend on the chosen sequence $\Lambda_k \uparrow \mathbb{Z}^d$.

Proof: We only prove the existence of μ^+ satisfying (i). The uniqueness is a direct consequence of Kolmogorov's theorem. The proof for μ^- is the same

Step.1: construction of μ^+.

Write $B_k = \{-k, \dots, k\}^d$. Let $A \subset\subset \mathbb{Z}^d$. Since n_A is increasing and local, the sequence $\left(\langle n_A \rangle^+_{B_k}\right)_{k \geq k_0}$ is well defined (provided k_0 large) and non increasing; We can define the decreasing limit:

$$\langle n_A \rangle^+ \underset{def}{:=} \lim_{k \to \infty} \langle n_A \rangle^+_{B_k}.$$

Now let $\Lambda \subset\subset \mathbb{Z}^d$ and $E \in \mathcal{F}_\Lambda$. Since $(n_A)_{A \subset \Lambda}$ is a basis of $\mathbb{R}^{\Omega_\Lambda}$, we can write $\mathbb{1}_E$ as a linear combination

$$\mathbb{1}_E = \sum_{A \subset \Lambda} \lambda_A n_A \quad \text{for } \lambda_A \in \mathbb{R}.$$

Then we define

$$\mu^+[E] \overset{def}{:=} \sum_{A \subset \Lambda} \lambda_A \langle n_A \rangle^+.$$

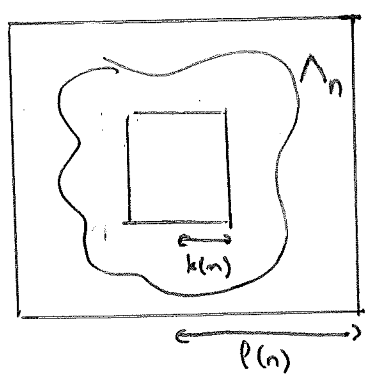
This way, we have defined $\mu^+ : \bigcup_{\Lambda \subset\subset \mathbb{Z}^d} \mathcal{F}_\Lambda \longrightarrow \mathbb{R}_+$, and $\forall \Lambda \subset\subset \mathbb{Z}^d$ $\mu|_{\mathcal{F}_\Lambda}$ probability measure (exercise).

By Kolmogorov's extension theorem, μ^+ can be extended into a probability measure on $\mathcal{F}$.

<u>Step. 2</u> Proof of (i). Let $\Lambda_n \uparrow \mathbb{Z}^d$. Let $(k(n))$ and $(\ell(n))$ be two sequences such that

$$\forall n \quad B_{k(n)} \subset \Lambda_n \subset B_{\ell(n)},$$

and $k(n), \ell(n) \xrightarrow[n \to \infty]{} \infty$.



By monotonicity, we have for every $A \subset\subset \mathbb{Z}^d$ and n large

$$\langle n_A \rangle^+_{B_{k(n)}} \geq \langle n_A \rangle^+_{\Lambda_n} \geq \langle n_A \rangle^+_{B_{\ell(n)}}$$

Hence $\lim_{n\to\infty} \langle n_A \rangle^+_{\Lambda_n} = \langle n_A \rangle^+$, which implies the result (since any local function can be writte as a linear combination of n_A's.) ∎

<u>To remember:</u> "μ^+ is the decreasing limit of $(\mu_{\Lambda_n})_n$"
in the sense $\langle f \rangle^+ = \lim_{n\to\infty} \downarrow \langle f \rangle^+_{\Lambda_n}$ $\forall f$ local $\uparrow$.

<u>Def:</u> A measure μ on $(\Omega, \mathcal{F})$ is said to be <u>translation invariant</u> if $\forall t \in \mathbb{Z}^d$

$$\theta_t \# \mu = \mu$$

where $\theta_t : \Omega \longrightarrow \Omega$ is defined by $\forall x \; (\theta_t \sigma)_x = \sigma_{x-t}$.

Equivalently, μ is translation invariant if
$\forall f$ meas. bounded $\int_\Omega f \, d\mu = \int_\Omega f \circ \theta_t^{-1} \, d\mu$

<u>Thm:</u> The measures μ^+ and μ^- are translation invariant.

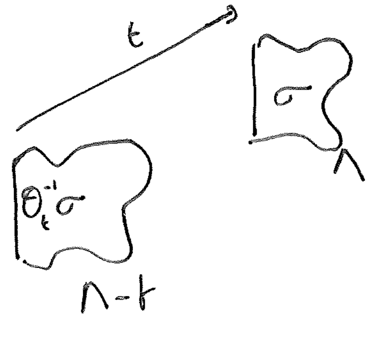
<u>Proof:</u> First, let $\Lambda \subset \mathbb{Z}^d$ and $\sigma \in \Omega_\Lambda$

$$H^+_{\Lambda-t}(\theta_t^{-1}(\sigma)) = -\beta \sum_{xy \in \overset{\circ}{E}_{\Lambda-t}} \sigma_{x+t}\sigma_{y+t} - \beta \sum_{\substack{xy \in E_{\Lambda-t} \\ y \in \partial(\Lambda-t)}} \sigma_{x+t} - h \sum_{x \in \Lambda-t} \sigma_x$$

$$= H^+_\Lambda(\sigma)$$

Hence $\forall \sigma \in \Omega_\Lambda$

$$\mu^+_{\Lambda-t}(\theta_t^{-1}\sigma) = \mu^+_\Lambda(\sigma) .$$



Now let E be a local event (i.e. $E \in \mathcal{F}_\Lambda$ for some $\Lambda \subset\subset \mathbb{Z}^d$).
Let $\Lambda_n \uparrow \mathbb{Z}^d$. For n large enough, the equation above implies

$$\underbrace{\mu^+_{\Lambda_n - t}\left(\theta_t^{-1} E\right)}_{\substack{\downarrow n\to\infty \\ \mu^+(\theta_t^{-1}E)}} = \underbrace{\mu^+_{\Lambda_n}(E)}_{\substack{\downarrow n\to\infty \\ \mu^+(E)}}$$

This concludes the proof because the local events generate the σ-algebra $\mathcal{F}$. ∎

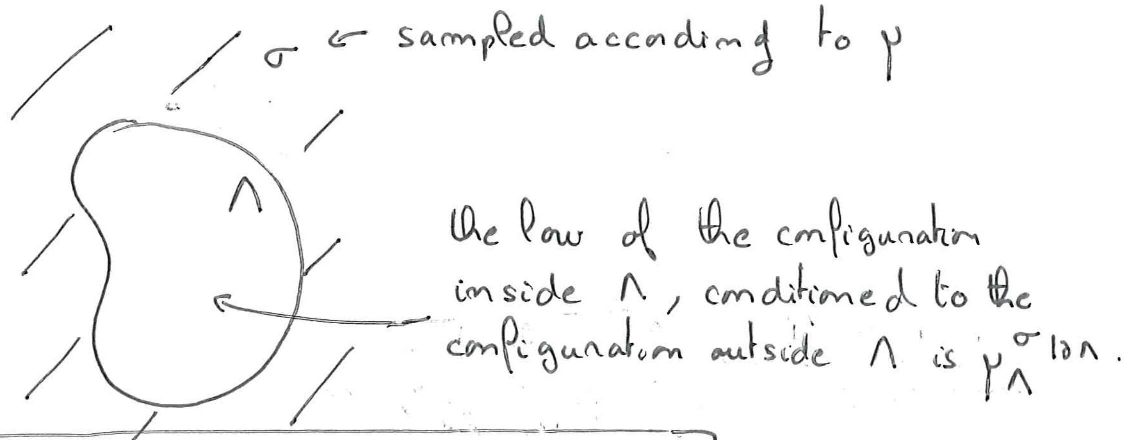
3 GENERAL INFINITE-VOLUME ISING MEASURE

Not: For $S \subset \mathbb{Z}^d$ (not necessarily finite), we write $\mathcal{F}_S = \sigma((\sigma_x)_{x\in S})$.

Def: A measure on $(\Omega, \mathcal{F})$ is called an infinite-volume Ising measure (or Gibbs state) at inverse temperature β and external field h, if for every $\Lambda \subset\subset \mathbb{Z}^d$ and f $\mathcal{F}_\Lambda$-measurable

$$\boxed{\langle f \mid \mathcal{F}_{\Lambda^c} \rangle (\sigma) = \langle f \rangle_\Lambda^{\sigma \mid \partial\Lambda} \quad \text{for } \mu\text{-a.e. } \sigma \in \Omega.}$$

conditional expectation of f w.r.t μ.



Thm: $\boxed{\mu^+ \text{ and } \mu^- \text{ are infinite-volume Ising measures}}$

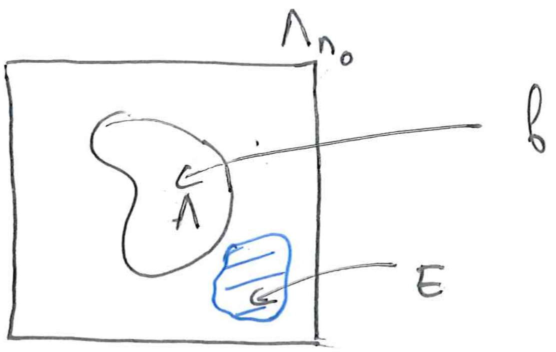
Proof: Let $\Lambda \subset\subset \mathbb{Z}^d$. Let f be a $\mathcal{F}_\Lambda$-measurable function.

We need to prove that for every $E \in \mathcal{F}_{\Lambda^c}$,

$$\langle f(\sigma) \cdot \mathbb{1}_E(\sigma) \rangle^+ = \langle \langle f \rangle_\Lambda^{\sigma \mid \partial\Lambda} \cdot \mathbb{1}_E(\sigma) \rangle^+$$

Since $E \in \mathcal{F}_{\Lambda^c}$ can be approximated by local events ($\forall \varepsilon > 0\ \exists E_{loc}$ local s.t. $\mu^+[E \triangle E_{loc}] < \varepsilon$), it suffices to prove the equation above for $E \in \mathcal{F}_{\Lambda^c}$ local.

Fix $E \in \mathcal{F}_{\Lambda^c}$ local. Let $\Lambda_n \uparrow \mathbb{Z}^d$. Let n_0 large enough s.t. $\Lambda \subset \Lambda_{n_0}$ and E is $\mathcal{F}_{\Lambda_{n_0}}$-measurable



By the domain Markov property, for every $n \geq n_0$

$$\forall \sigma \in \Omega_{\Lambda_n} \quad \langle f \mid \mathcal{F}_{\Lambda_n \setminus \Lambda} \rangle^+_{\Lambda_n}(\sigma) = \langle f \rangle^{\sigma \mid \partial\Lambda}_{\Lambda}.$$

Therefore

$$\langle f \mathbb{1}_E \rangle^+_{\Lambda_n} = \langle \langle f \mathbb{1}_E \mid \mathcal{F}_{\Lambda_n \setminus \Lambda} \rangle^+_{\Lambda_n} \rangle^+_{\Lambda_n}$$

$$= \langle \langle f \mid \mathcal{F}_{\Lambda_n \setminus \Lambda} \rangle^+_{\Lambda_n} \mathbb{1}_E \rangle_{\Lambda_n}$$

$$= \langle \langle f \rangle^{\sigma \mid \partial\Lambda}_{\Lambda} \mathbb{1}_E \rangle^+_{\Lambda_n}.$$

Since f, $\mathbb{1}_E$ and $\sigma \longmapsto \langle f \rangle^{\sigma \mid \partial\Lambda}_{\Lambda}$ are local, we can take the limit ~~as~~ n tends to infinity in the equation above, which concludes the proof. ■

Prop: Let μ be an infinite-volume Ising measure. Then for every local fction f ↑,

$$\langle f \rangle^- \leq \underbrace{\langle f \rangle}_{\text{"expectation of } f \text{ w.r.t } \mu\text{"}} \leq \langle f \rangle^+ .$$

Proof: Let $\Lambda_n \uparrow \mathbb{Z}^d$. For n large enough, and $\forall\, \omega \in \{-1,1\}^{\partial\Lambda}$

$$\langle f \rangle^-_{\Lambda_n} \leq \langle f \rangle^\omega_{\Lambda_n} \leq \langle f \rangle^+_{\Lambda_n}$$

Therefore, we have

$$\langle f \rangle^-_{\Lambda_n} \leq \langle f \mid \mathcal{F}_{\Lambda_n^c} \rangle \leq \langle f \rangle^+_{\Lambda_n} \quad \mu\text{-a.s.}$$

Taking the expectation w.r.t. μ and letting n tend to infinity concludes the proof. ∎

Application: if $\mu^- = \mu^+$, then there is a unique infinite-volume Ising measure.

Thm [First characterization of uniqueness]

For fixed $\beta \geq 0$ $h \in \mathbb{R}$, the following are equivalent.

(i) there exists a unique infinite-Volume Ising measure

(ii) $\mu^- = \mu^+$.

(iii) $\langle \sigma_0 \rangle^+ = \langle \sigma_0 \rangle^-$.

Proof: (ii) $\Leftrightarrow$ (i) follows from the proposition above.

(ii) $\Rightarrow$ (iii) trivial.

(iii) $\Rightarrow$ (ii) Let $A \subset\subset \mathbb{Z}^d$. Since the function

$$\sum_{x \in A} n_x - n_A$$

is increasing, we have

$$\langle \sum_{x \in A} n_x - n_A \rangle^- \leq \langle \sum_{x \in A} n_x - n_A \rangle^+.$$

Therefore,

$$\langle n_A \rangle^+ - \langle n_A \rangle^- \leq \sum_{x \in A} \langle n_x \rangle^+ - \langle n_x \rangle^-$$

$$\underset{\substack{\uparrow \\ \text{translation invariance}}}{=} \frac{|A|}{2} \left(\langle \sigma_0 \rangle^+ - \langle \sigma_0 \rangle^- \right).$$

If $\langle \sigma_0 \rangle^- = \langle \sigma_0 \rangle^+$ then $\forall A \subset\subset \mathbb{Z}^d$ $\langle n_A \rangle^- = \langle n_A \rangle^+$, which implie $\mu^- = \mu^+$ ∎

4 MAGNETIZATION.

Not: $m(\beta, h) = \langle \sigma_0 \rangle^+_{\beta,h} = \lim_{n \to \infty} \langle \sigma_0 \rangle^+_{\Lambda_n, \beta, h}$

"magnetization at (β, h)"

Prop: Fix $\beta \geq 0$.

The function $m(\beta, \cdot)$ is right-continuous, nondecreasing on $\mathbb{R}$.

Proof: For every $n \geq 1$, define $f_n(h) = \langle \sigma_0 \rangle^{+}_{\Lambda_n, \beta, h}$.

Then $m(\beta, \cdot) = \operatorname{Inf}_{n \in \mathbb{N}} f_n$

Since f_n is non decreasing upper semi-continuous (it is continuous).

$m(\beta, \cdot)$ is also nondecreasing and upper semi-continuous and therefore it is also right continuous.

$$\left(\text{indeed } m(\beta,h) \underset{\substack{\uparrow \\ \text{"monotonicity"}}}{\leq} \liminf_{h' \downarrow h} m(\beta, h') \leq \limsup_{h' \downarrow h} m(\beta,h') \underset{\substack{\uparrow \\ \text{"u.s.c"}}}{\leq} m(\beta,h) \right)$$

■

Rk: $h \longmapsto \langle \sigma_0 \rangle^{-}_{\beta,h}$ is nondecreasing, left-continuous.

$\left(\text{indeed } \langle \sigma_0 \rangle^{-}_{\beta,h} = - m(\beta, -h) \right)$

Exercise:

Let $B_n = \{-n, \dots, n\}^d$. Prove that

$$m(\beta, h) = \lim_{n \to \infty} \frac{1}{|B_n|} \sum_{x \in B_n} \langle \sigma_x \rangle^{+}_{B_n}.$$

Thm:

Let $\beta \geq 0$ $h \in \mathbb{R}$. The following are equivalent.

(i) There exists a unique infinite volume Ising measure.

(ii) $m(\beta, \cdot)$ is continuous at h.

Lemma: $\forall f : \Omega_\Lambda \to \mathbb{R}$, we have

$$\langle f \rangle^+_{\Lambda,h} = \frac{\langle f\, e^{(h-h')S + 2\beta S'} \rangle^-_{\Lambda,h'}}{\langle e^{(h-h')S + 2\beta S'} \rangle^-_{\Lambda,h'}}$$

where $S(\sigma) = \sum_{x\in\Lambda} \sigma_x \qquad S'(\sigma) = \sum_{\substack{xy\in E \\ y\in\partial\Lambda}} \sigma_x$.

Proof: $\mu^+_{\Lambda,h'}(\sigma) = e^{(h'-h)S + 2\beta S'} \mu^-_{\Lambda,h}(\sigma)$.

Hence $Z^+_{\Lambda,h'}[f] = Z^-_{\Lambda,h}[f e^{(h'-h)S + 2\beta S'}]$

$Z^+_{\Lambda,h'} = Z^-_{\Lambda,h}[e^{(h'-h)S + 2\beta S'}]$ $\blacksquare$

Proof of the theorem:

It suffices to prove that

$$\lim_{h'\uparrow h} \langle \sigma_0 \rangle^+_{h'} = \langle \sigma_0 \rangle^-_h \qquad (*)$$

Indeed $(ii) \Leftrightarrow \lim_{h'\uparrow h} \langle \sigma_0 \rangle^+_{h'} = \langle \sigma_0 \rangle^+_h$ $\overset{(*)}{\Leftrightarrow}$ $\langle \sigma_0 \rangle^-_h = \langle \sigma_0 \rangle^+_h$

$\Leftrightarrow (i)$.

$\boxed{\geq}$ $\forall h' < h$ we have $\langle \sigma_0 \rangle^+_{h'} \geq \langle \sigma_0 \rangle^-_{h'}$.

and therefore $\lim_{h'\uparrow h} \langle \sigma_0 \rangle^+_{h'} \geq \langle \sigma_0 \rangle^-_h$

by left continuity in h of $\langle \sigma_0 \rangle^-_h$.

$\boxed{\leq}$ Let $h' < h$, we prove that

$$\underbrace{\langle \sigma_0 \rangle^+_{h'}}_{a:=} \leq \underbrace{\langle \sigma_0 \rangle^-_{h}}_{b:=}$$

Let $n \geq 1$. Write $S = \sum_{x \in B_n} \sigma_x$ $\quad S' = \sum_{\substack{xy \in E_{B_n} \\ y \in \partial B_n}} \sigma_x$

$$\langle S \rangle^+_{B_n, h'} = \sum_{x \in B_n} \langle \sigma_x \rangle^+_{B_n, h'} \underset{\substack{\uparrow \\ \text{monotonicity in} \\ \text{the graph}}}{\geq} |B_n| \, a.$$

Let $\varepsilon > 0$ small.

$$\mu^+_{B_n, h'} [S \leq (a-\varepsilon)|B_n|] = \mu [|B_n| - S \geq (1-a+\varepsilon) |B_n|]$$

$$\overset{\text{Markov}}{\leq} \frac{1-a}{1-a+\varepsilon} \leq 1 - \frac{\varepsilon}{2}$$

Equivalently.

$$\langle S \rangle^-_{B_n, h} = \sum_{x \in B} \langle \sigma_x \rangle^-_{B_n, h} \leq |B_n| \times b$$

Hence

$$\mu^-_{B_n, h} [S \geq (b+\varepsilon) |B_n|] \leq \frac{b}{b+\varepsilon} \leq 1 - \frac{\varepsilon}{2}.$$

$$\frac{\varepsilon}{2} \leq \mu^+_{B_n, h'} [S \geq (a-\varepsilon) |B_n|] = \frac{\langle \mathbb{1}_{S \geq (a-\varepsilon)|B_n|} \, e^{(h'-h)S + 2\beta S'} \rangle^-_{B_n, h}}{\langle e^{(h'-h)S + 2\beta S'} \rangle^-_{B_n, h}}$$

$$\underset{\substack{\uparrow \\ |S'| \leq d |\partial \Lambda_n|}}{\leq} \frac{e^{(h'-h)(a-\varepsilon)|B_n| + 4\beta d |\partial \Lambda_n|}}{\langle \mathbb{1}_{S \leq (b+\varepsilon)|B_n|} \, e^{(h'-h)S} \rangle^-_{B_n, h}}$$

hence $\forall n$ $\quad \left(\frac{\varepsilon}{2}\right)^2 \leq e^{(h'-h)(a-b-2\varepsilon)|B_n|} \times e^{4\beta|\partial B_n|}$

Since $|\partial B_n| \ll |B_n|$, we must have

$$a - b - 2\varepsilon \leq 0$$

■

Theorem:

$\forall h \neq 0$, there exists a unique infinite volume Ising measure.

The proof relies on the GHS inequality (Griffiths, Hurst, Sherman)

Prop: [GHS-inequality]

Let $\Lambda \subset\subset \mathbb{Z}^d$ $\forall x, y, z \in \Lambda$ $\quad \forall h \geq 0$

$$\langle \sigma_x ; \sigma_y ; \sigma_z \rangle^+_{\Lambda, h} \leq 0$$

where $\langle \sigma_x ; \sigma_y ; \sigma_z \rangle^+ = \langle \sigma_x \sigma_y \sigma_z \rangle - \langle \sigma_x \rangle \langle \sigma_y \sigma_z \rangle - \langle \sigma_y \rangle \langle \sigma_x \sigma_z \rangle$
$\qquad - \langle \sigma_z \rangle \langle \sigma_x \sigma_y \rangle + 2 \langle \sigma_x \rangle \langle \sigma_y \rangle \langle \sigma_z \rangle$

Proof: later.

Proof of the Theorem:

We prove that $\forall \Lambda$ $\langle \sigma_0 \rangle^+_{\Lambda, h}$ is a concave function of $h \in [0, \infty)$. Hence $m(\beta, \cdot)$ is concave on $[0, \infty)$ (as a simple limit of concave functions). Therefore it is continuous on $(0, \infty)$ which implies the theorem for $h > 0$.

Since $\langle \sigma_0 \rangle^-_h = \langle -\sigma_0 \rangle^+_{-h}$ and $\langle \sigma_0 \rangle^+_{+h} = \langle -\sigma_0 \rangle^-_{-h}$

we also have $\langle \sigma_0 \rangle^-_h = \langle \sigma_0 \rangle^+_h$ for $h < 0$.

Let $\Lambda \subset\subset \mathbb{Z}^d$. For every $A \subset \Lambda$

$$\frac{d}{dh} Z^+_{\Lambda,h}[\sigma_A] = \sum_{\sigma \in \Omega_\Lambda} \sigma_A \cdot \sum_{x \in \Lambda} \sigma_x \cdot e^{H^+_{\Lambda,h}(\sigma)}$$

$$= \sum_{x \in \Lambda} Z^+[\sigma_x \sigma_A]$$

Hence

$$\frac{d}{dh} \langle \sigma_A \rangle^+_{\Lambda,h} = \frac{d}{dh} \left(\frac{Z^+_{\Lambda,h}[\sigma_A]}{Z^+_{\Lambda,h}[1]} \right)$$

$$= \frac{\sum_{x \in \Lambda} Z^+_{\Lambda,h}[\sigma_x \sigma_A] \cdot Z^+_{\Lambda,h} - \sum_{x \in \Lambda} Z^+[\sigma_A] Z^+[\sigma_x]}{{Z^+_{\Lambda,h}}^2}$$

$$= \sum_{x \in \Lambda} \langle \sigma_x \sigma_A \rangle^+_{\Lambda,h} - \langle \sigma_x \rangle^+_{\Lambda,h} \langle \sigma_A \rangle^+_{\Lambda,h}$$

Hence defining $g(h) = \langle \sigma_0 \rangle^+_{\Lambda,h}$, we see that

$$g'(h) = \sum_{x \in \Lambda} \langle \sigma_0 \sigma_x \rangle^+_{\Lambda,h} - \langle \sigma_0 \rangle^+_{\Lambda,h} \langle \sigma_x \rangle^+_{\Lambda,h}$$

$$g''(h) = \sum_{x,y \in \Lambda} \langle \sigma_0 ; \sigma_x ; \sigma_y \rangle^+_{\Lambda,h} \underset{\substack{\uparrow \\ \text{GHS}}}{\leq} 0 \quad \text{if } h \geq 0$$

CHAPTER 6 :

PRESSURE.

$\beta \geq 0$ fixed. $\omega \in \{0, +1, -1\}^{\partial\Lambda}$.

$$H^{\omega}_{\Lambda,h}(\sigma) = -\beta \sum_{xy \in \overset{\circ}{E}} \sigma_x \sigma_y - \beta \sum_{\substack{xy \in E \\ y \in \partial\Lambda}} \sigma_x \omega_y - h \sum_x \sigma_x.$$

$$= H^{\omega}_{\Lambda,0}(\sigma) - h \sum_{x \in \Lambda} \sigma_x$$

$$H^{0}_{\Lambda,h}(\sigma) = -\sum_{xy \in \overset{\circ}{E}} \sigma_x \sigma_y - h \sum_{x \in \Lambda} \sigma_x$$

1 DEFINITION OF THE PRESSURE.

Notation: For $\Lambda \subset\subset \mathbb{Z}^d$, $\omega \in \{-1, 0, 1\}^{\partial\Lambda}$

$$f^{\omega}_{\Lambda}(h) := \frac{1}{|\Lambda|} \log\left(Z^{\omega}_{\Lambda,h}\right)$$

Thm: Let $B_n = \{-2^n, \dots, 2^n - 1\}$, $h \in \mathbb{R}$.
$\forall$ seq. $\omega_n \in \{-1, 0, 1\}^{\partial B_n}$, $\left(f^{\omega_n}_{B_n}(h)\right)$ converges and

$$f(h) = \lim_{n \to \infty} f^{\omega}_{B_n}(h) \qquad \text{"pressure"}$$

does not depend on (ω_n).

Rk: $Z^{\omega}_{B_n,h} = e^{f(h)|B_n| + o(|B_n|)}$

$\rightarrow$ the choice of B_n is important here because $\frac{|\partial B_n|}{|B_n|} \rightarrow 0$.

Exercises.

- Prove that $\forall\ \Lambda_n \uparrow \mathbb{Z}^d$ s.t. $\frac{|\partial \Lambda_n|}{|\Lambda_n|} \to 0$

$$\lim_{n \to \infty} f_{\Lambda_n}^{\omega_n}(h) = f(h)$$

- Give an example of $\Lambda_n \uparrow \mathbb{Z}^d$ and (ω_n) b.c. s.t.

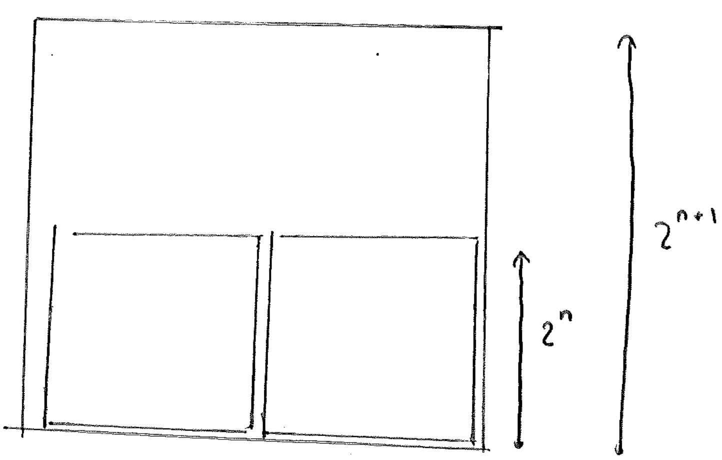
$$f_{\Lambda_n}^{\omega_n}(h) \underset{n \to \infty}{\not\to} f(h).$$

Proof of the theorem.

We begin with the free b.c. $\omega = 0$.

Let $n \geq 1$ and consider a covering of B_{n+1} with 2^d translated copies of B_n. Namely we consider $B^{(1)}, \dots, B^{(2^d)}$ disjoint translates of B_n s.t.

$$B_{n+1} = B^{(1)} \cup \dots \cup B^{(2^d)}$$



For $\sigma \in \Omega_{B_{n+1}}$ write $\sigma^{(k)} = \sigma_{|B^{(k)}}$

$$H^0_{B_{n+1},h}(\sigma) = \sum_{k=1}^{2^d} H^0_{B^{(k)},h}(\sigma^{(k)}) + S_n(\sigma)$$

where $|S_n| \leq C\,|\partial B_n|$ "contribution of the boundary edges.

Hence

$$\overset{\circ}{Z}_{B_{n+1},h} = \sum_{\sigma\in\Omega_{B_{n+1}}} e^{-\overset{\circ}{H}_{B_{n+1},h}(\sigma)}$$

$$= \sum_{\sigma^{(1)}} \cdots \sum_{\sigma^{(2^d)}} e^{-H^{\circ}_{B^{(1)},h}(\sigma^{(1)}) - \cdots - H^{\circ}_{B^{(2^d)},h}(\sigma^{(2^d)}) + \delta_n(\sigma)}$$

$$\leq e^{c|\partial B_n|}\left(\overset{\circ}{Z}_{B_n,h}\right)^{2^d}$$

Therefore

$$\overset{\circ}{f}_{B_{n+1}}(h) = \frac{1}{|B_{n+1}|}\log\left(\overset{\circ}{Z}_{B_{n+1},h}\right) \leq \frac{1}{2^d|B_n|}\times 2^d\log\left(\overset{\circ}{Z}_{B_n,h}\right) + c\frac{|\partial B_n|}{|B_n|}$$

$$= \overset{\circ}{f}_{B_n}(h) + c\frac{|\partial B_n|}{|B_n|}.$$

Equivalently $\overset{\circ}{f}_{B_{n+1}}(h) \geq \overset{\circ}{f}_{B_n}(h) - c\dfrac{|\partial B_n|}{|B_n|}$

Since $\left|\dfrac{\partial B_n}{B_n}\right| = O\left(\dfrac{1}{2^n}\right)$, $\left(\overset{\circ}{f}_{B_n}(h)\right)_n$ is a Cauchy sequence,

and therefore converges to a limit $f(h)$.

Now for every sequence (ω_n) of b.c., we have

$$\left|\overset{\circ}{H}_{B_n,h}(\sigma) - H^{\omega_n}_{B_n,h}(\sigma)\right| \leq d\,|\partial B_n|$$

Hence $\left|\overset{\circ}{f}_{B_n}(h) - f^{\omega_n}_{B_n}(h)\right| \leq d\,\dfrac{|\partial B_n|}{|B_n|}$,

and therefore $f^{\omega_n}_{B_n}(h) \xrightarrow[n\to\infty]{} f(h)$

Prop: Analytic properties of the pressure.

The function $f : \mathbb{R} \to \mathbb{R}$ is convex and

$$\forall h \quad \frac{\partial f}{\partial h^+}(h) = \langle \sigma_0 \rangle_h^+ \qquad \frac{\partial f}{\partial h^-}(h) = \langle \sigma_0 \rangle_h^- .$$

Proof: Since a pointwise limit of convex fcts is convex it suffices to prove that $\forall n$ $f^0_{B_n}$ is convex.

Consider the measure λ on Ω_{B_n} defined by

$$\lambda(\{\sigma\}) = e^{\beta \sum_{xy \in E^0} \sigma_x \sigma_y}$$

and $$S(\sigma) = \sum_{x \in B_n} \sigma_x$$

Then $\forall h \in \mathbb{R}$ $$Z^0_{B_n,h} = \int_{\Omega_{B_n}} e^{hS} \, d\lambda$$

Hence $\forall$ $h, h' \in \mathbb{R}$ $\alpha \in [0,1]$

$$Z^0_{B_n, \alpha h + (1-\alpha) h'} = \int e^{\alpha h S} \cdot e^{(1-\alpha) h' S} \, d\lambda$$

$$\underset{p = \frac{1}{\alpha},\ q = \frac{1}{1-\alpha}}{\overset{\text{Hölder}}{\leq}} \left(\int e^{hS} d\lambda\right)^{\alpha} \cdot \left(\int e^{h'S} d\lambda\right)^{1-\alpha}$$

$$= \left(Z^0_{B_n, h}\right)^{\alpha} \cdot \left(Z_{B_n, h'}\right)^{1-\alpha}$$

taking the logarithm and dividing by $|B_n|$, we get

$$f^0_{B_n}(\alpha h + (1-\alpha) h') \leq \alpha f^0_{B_n}(h) + (1-\alpha) f^0_{B_n}(h').$$

It remains to compute the left and right derivatives.

Fix $n \in \mathbb{N}$, For every w b.c. for B_n, we have

$$\frac{d}{dh}\left(Z^{w}_{B_n,h}\right) = \sum_{\sigma \in \Omega_{B_n}} \left(\sum_{x \in B_n} \sigma_x\right) e^{-H^{w}_{B_n,h}(\sigma)}$$

Therefore

$$\frac{d}{dh} f^{w}_{B_n}(h) = \frac{1}{|B_n|} \left\langle \sum_{x \in B_n} \sigma_x \right\rangle^{w}_{B_n,h}.$$

Fix $h_0 \in \mathbb{R}$. Since $\langle \sigma_x \rangle^{+}_{B_n,h}$ is non decreasing in h, the mean-value theorem implies that

$$\forall h > h_0 \quad \frac{1}{|B_n|} \sum_{x \in B_n} \langle \sigma_x \rangle^{w}_{B_n,h_0} \leq \frac{f^{w}_{B_n}(h) - f^{w}_{B_n}(h_0)}{h - h_0} \leq \frac{1}{|B_n|} \sum_{x \in B_n} \langle \sigma_x \rangle^{w}_{B_n,h}$$

Applying it to $w = +$ and to $w = -$, and using comparison between b.c. we get $\forall h > h_0$

$$\langle \sigma_0 \rangle^{+}_{h_0} \leq \frac{f^{+}_{B_n}(h) - f^{+}_{B_n}(h_0)}{h - h_0}$$

and
$$\frac{f^{-}_{B_n}(h) - f^{-}_{B_n}(h_0)}{h - h_0} \leq \langle \sigma_0 \rangle^{-}_{h} \leq \langle \sigma_0 \rangle^{+}_{h}$$

Letting n tend to infinity, we get

$$\forall h > h_0 \quad \langle \sigma_0 \rangle^{+}_{h_0} \leq \frac{f(h) - f(h_0)}{h - h_0} \leq \langle \sigma_0 \rangle^{+}_{h}$$

and the proof follows from the right continuity of $\langle \sigma_0 \rangle^{+}_{h}$ in h. Equivalently, the left derivative is computed using the left-continuity of $\langle \sigma_0 \rangle^{-}_{h}$ in h.

Corollary: Are equivalent.

(i) there exists a unique infinite-volume Ising measure

(ii) f is differentiable at h.

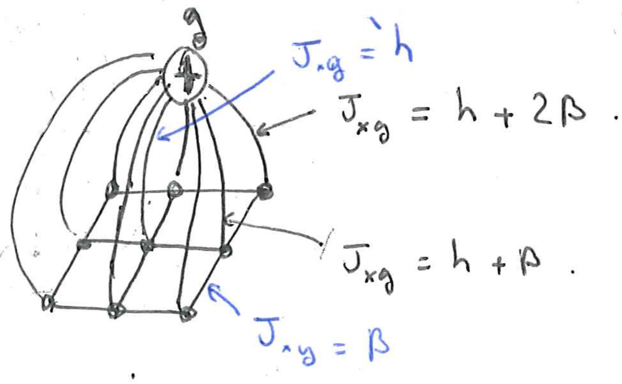
CHAPTER 7

RANDOM CURRENTS

- $G = (V, E)$ finite graph $\quad g \in V \quad$ "ghost"
- Ising configuration $\Omega = \{\sigma \in \{+1, -1\}^V \text{ s.t. } \sigma_g = +1\}$.
- $(J_{xy})_{xy \in E}$ coupling constants $\quad J_{xy} \geq 0$
- $H(\sigma) = - \sum_{xy \in E} J_{xy} \sigma_x \sigma_y \quad \longrightarrow \quad \mu(\sigma) = \frac{1}{Z} e^{-H(\sigma)}$

Rk: Ising in Λ with n.n. interactions, inv. temperature β, ext. field $h \geq 0$, + b.c. fits in this framework.

$$V = \Lambda \cup \{g\} \qquad E = \{xy,\ x, y \in \Lambda\} \cup \{xg,\ x \in \Lambda\}$$



$J_{xy} = \beta$ for $x, y \in \Lambda \quad x \sim y$

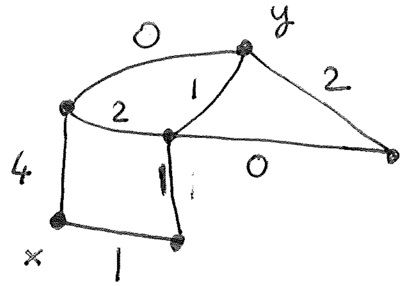
$$J_{xg} = \underset{\substack{\uparrow \\ \text{ext field}}}{h} + \underset{\substack{\uparrow \\ + \text{ b.c.}}}{\sum_{\substack{y \sim x \\ y \in \partial\Lambda}} \beta}$$

1 RANDOM CURRENT REPRESENTATION

Def: We call current on G a function

$$\mathbf{n} : E \longrightarrow \mathbb{N}$$

- sources of n $\quad \partial \mathbf{n} = \{ x \in V : \sum_{e \ni x} n_e \text{ odd} \}$
- $x \overset{n}{\longleftrightarrow} y$ if there exists a path γ from x to y with $n_e > 0 \ \forall e \in \gamma$.



a current with $\partial n = \{x, y\}$

Rk: $\sum_x \sum_{e \ni x} n_e = 2 \sum_e n_e$

$\rightarrow |\partial n|$ is always even.

Ex: If $\partial n = \{x, y\}$ then $x \overset{n}{\longleftrightarrow} y$.

Thm: (random current representation of Ising)

Let $A \subset V$ even.

$$\langle \sigma_A \rangle = \frac{\sum_{\partial n = A} w(n)}{\sum_{\partial n = \emptyset} w(n)} \quad \text{where } w(n) = \prod_{e \in E} \frac{J_e^{n_e}}{n_e!}$$

Lemma: Let I finite, J finite or countable.
$a_{i,j} \in \mathbb{R} \ \forall i \in I \ j \in J$. Assume $\sum_{j \in J^I} \prod_{i \in I} |a_{i j_i}| < \infty$

Then $\prod_{i \in I} \left(\sum_{j \in J} a_{ij} \right) = \sum_{j \in J^I} \prod_{i \in I} a_{i j_i}$

<u>Proof</u>: exercise : use Fubini.

<u>Proof of Thm</u>:

$$Z[\sigma_A] = \sum_{\sigma\in\Omega} \sigma_A \, e^{\sum_{xy\in E} J_{xy}\sigma_x\sigma_y}$$

$$= \sum_{\sigma\in\Omega} \sigma_A \prod_{xy\in E} \left(\sum_{n\in\mathbb{N}} \frac{1}{n!} (J_{xy}\sigma_x\sigma_y)^n \right)$$

$$\overset{\text{Lemma}}{=} \sum_{\sigma\in\Omega} \sigma_A \sum_{n\in E^{\mathbb{N}}} \prod_{xy\in E} \frac{1}{n_{xy}!} (J_{xy}\sigma_x\sigma_y)^{n_{xy}}$$

$$\overset{\text{Fubini}}{=} \sum_{n} w(n) \sum_{\sigma\in\Omega} \underbrace{\sigma_A \prod_{xy\in E} (\sigma_x\sigma_y)^{n_{xy}}}_{= \prod_{x\in V} \sigma_x^{\mathbb{1}_{x\in A} + \sum_{e\ni x} n_e}}$$

$$= |\Omega| \sum_{n} w(n) \underbrace{\frac{1}{|\Omega|} \sum_{\sigma\in\Omega} \left(\prod_{x\in V} \sigma_x^{\mathbb{1}_{x\in A} + \mathbb{1}_{x\in\partial n}} \right)}_{= \mathbb{1}_{\partial n = A}}$$

$$= |\Omega| \cdot \sum_{\partial n = A} w(n)$$ ∎

<u>Rk</u>: If $A \subset V$

$$\langle \sigma_A \rangle = \langle \sigma_{A \Delta \{g\}} \rangle = \frac{\sum_{\partial n = A\Delta\{g\}} w(n)}{\sum_{\partial n = \emptyset} w(n)}$$

2. SWITCHING LEMMA.

Prop: (Switching lemma)

Let $F : \mathbb{N}^E \longrightarrow \mathbb{R}$ (≥ 0 a bounded) $x, y \in V$ $A \subset V$

Then

$$\sum_{\substack{\partial m = A \\ \partial n = \{x,y\}}} w(m)\, w(n)\, F(m+n) = \sum_{\substack{\partial m = A \Delta \{x,y\} \\ \partial n = \emptyset}} w(m)\, w(n)\, F(m+n)\, \mathbb{1}_{x \overset{m+n}{\longleftrightarrow} y}$$

Application:

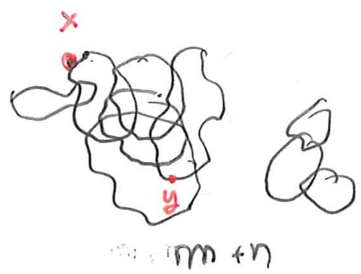
$$\left(\text{Not. } Z_c = \sum_{\partial n = \emptyset} w(n) = \frac{1}{|\Omega|} Z \right)$$

$$\langle \sigma_x \sigma_y \rangle^2 = \frac{1}{Z_c^2} \sum_{\substack{\partial m = \{x,y\} \\ \partial n = \{x,y\}}} w(m) \cdot w(n)$$

$$= \frac{1}{Z_c^2} \sum_{\substack{\partial m = \emptyset \\ \partial n = \emptyset}} w(m)\, w(n)\, \mathbb{1}_{x \overset{m+n}{\longleftrightarrow} y}$$

$$= \frac{\sum_{\substack{\partial m = \emptyset \\ \partial n = \emptyset}} w(m)\, w(n)\, \mathbb{1}_{x \overset{m+n}{\longleftrightarrow} y}}{\sum_{\substack{\partial m = \emptyset \\ \partial n = \emptyset}} w(m)\, w(n)}.$$

$$= P\left[x \overset{m+n}{\longleftrightarrow} y \right]$$



where $P[(m,n)] = \frac{1}{Z_c^2} w(m)\, w(n)$

<u>Notation</u>: Let k, n be two currents s.t. $n \leq k$ (ie $\forall e\ n_e \leq k_e$)

$$\binom{k}{n} := \prod_{e \in E} \binom{k_e}{n_e}$$

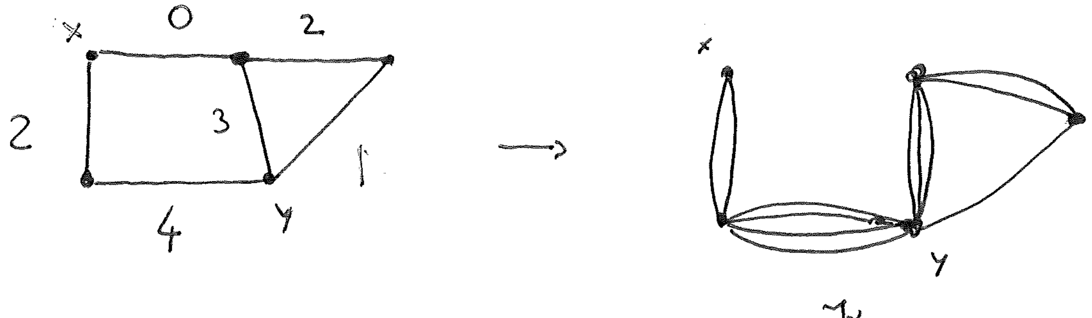
<u>Lemma</u>:

Fix k current s.t. $x \xleftrightarrow{k} y$. Then

$$\sum_{\substack{n \leq k \\ \partial n = xy}} \binom{k}{n} = \sum_{\substack{n \leq k \\ \partial n = \emptyset}} \binom{k}{n}$$

<u>Rk</u>: $G =$ — $\displaystyle\sum_{\substack{n \leq k \\ n \text{ odd}}} \binom{k}{n} = \sum_{\substack{n \leq k \\ n \text{ even}}} \binom{k}{n}$

<u>Proof</u>: Let $\mathcal{K}$ be the graph with vertex set V. and k_{xy} parallel edges between x and y



$\mathcal{K}$

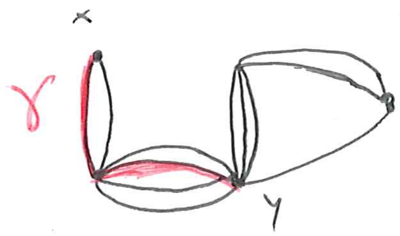
If $n \leq k$ then $\binom{k}{n}$ is the number of subgraphs of $\mathcal{K}$ ($\mathcal{J} \subset \mathcal{K}$) with exactly n_{xy} edges between x and y.

Hence for $A \subset V$

$$\sum_{\substack{n \leq k \\ \partial n = A}} \binom{k}{n} = \left| \{ \mathcal{N} \subset \mathcal{G}_k : \partial \mathcal{N} = A \} \right|$$

where $\partial \mathcal{N} = A$ means that the vertices of A have odd degrees in $\mathcal{N}$, the other vertices have even degrees.

Fix a path γ from x to y in $\mathcal{G}_k$



symmetric difference
$\forall e \in \gamma \quad e \in \mathcal{N} \iff e \notin \mathcal{N} \Delta \gamma$

Then $\mathcal{N} \longrightarrow \mathcal{N} \Delta \gamma$ is a bijection from $\{ \mathcal{N} \subset \mathcal{G}_k : \partial \mathcal{N} = xy \}$ to $\{ \mathcal{N} \subset \mathcal{G}_k : \partial \mathcal{N} = \emptyset \}$ (indeed $(\mathcal{N} \Delta \gamma) \Delta \gamma = \mathcal{N} \ \forall \mathcal{N}$), and therefore

$$\sum_{\substack{n \leq k \\ \partial n = xy}} \binom{k}{n} = \sum_{\substack{n \leq k \\ \partial n = \emptyset}} \binom{k}{n} \qquad \blacksquare$$

<u>Ext.</u> If $k \in \mathcal{F}_A := \{ n :$ every connected component* of n intersects A at an even number of vertices $\}$

$$\sum_{\substack{n \leq k \\ \partial n = A}} \binom{k}{n} = \sum_{\substack{n \leq k \\ \partial n = 0}} \binom{k}{n}$$

* for the graph with edge set $\{ e : n_e > 0 \}$.

Proof of the switching lemma.

$$\sum_{\substack{\partial m = A\\ \partial n = xy}} w(m)\,w(n)\,F(m+n) = \sum_{\substack{\partial k = A\Delta xy\\ \partial n = xy\\ n\le k}} w(k-n)\,w(n)\,F(k)\,\mathbb{1}_{x\overset{k}{\longleftrightarrow} y}$$

Change of variable

$m, n \longrightarrow (m+n, n)$

$$= \sum_{\partial k = A\Delta xy} w(k)\,F(k) \sum_{\substack{\partial n = xy\\ n\le k}} \underbrace{\frac{w(k-n)\,w(n)}{w(k)}}_{=\binom{k}{n}} \cdot \mathbb{1}_{x\overset{k}{\longleftrightarrow} y}$$

Lemma

$$= \sum_{\partial k = A\Delta xy} w(k)\,F(k)\,\mathbb{1}_{x\overset{k}{\longleftrightarrow} y} \sum_{\substack{\partial n = \emptyset\\ n\le k}} \binom{k}{n}$$

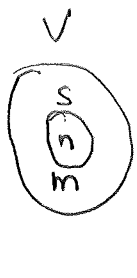
$$= \sum_{\substack{\partial m = A\Delta xy\\ \partial n = \emptyset}} w(m)\,w(n)\,F(m+n)\,\mathbb{1}_{x\overset{m+n}{\longleftrightarrow} y} \qquad \blacksquare$$

$(k,n) \longmapsto (k-n, n)$

Generalizations:

(i)
$$\sum_{\substack{\partial m = A\\ \partial n = B}} w(m)\,w(n)\,F(m+n) = \sum_{\substack{\partial m = A\Delta B\\ \partial n = \emptyset}} w(m)\,w(n)\,F(m+n)\,\mathbb{1}_{m+n\in\mathcal{F}_B}$$

(ii) below n denote a current in the subgraph induced by $S\subset V$. $w_S(n)$ is the associated weight. m is a current on V. $x,y\in S$



$$\sum_{\substack{\partial m = A\\ \partial n = xy}} w(m)\,w_S(n) = \sum_{\substack{\partial m = A\Delta xy\\ \partial n = \emptyset}} w(m)\,w_S(n)\,\mathbb{1}_{x\overset{m|_S+n}{\longleftrightarrow} y}$$

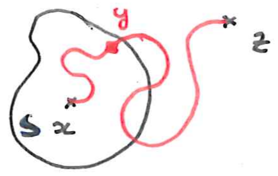
3. SIMON LIEB. INEQUALITY.

Thm: [Simon-Lieb inequality.]

Let $S \subset V$ $\quad x \in S$ $\quad z \in V \setminus S$ $\quad$ (Not. $\partial_{in} S = \{u \in S : \exists v \in V\setminus S \ u \sim v\}$)

$$\langle \sigma_x \sigma_z \rangle \leq \sum_{y \in \partial_{in} S} \langle \sigma_x \sigma_y \rangle_S \langle \sigma_y \sigma_z \rangle$$

Proof: Let m current with $\partial m = xz$.



Then $\exists\, y \in \partial^{in} S$ s.t. $x \xleftrightarrow{m|_S} y$

$$Z_V \langle \sigma_x \sigma_z \rangle Z_S = \sum_{\substack{\partial m = xz \\ \partial n_S = \emptyset}} w(m)\, w(n_S)$$

"current in S" $\rightarrow$ n_S

$$\leq \sum_{y \in \partial_{in} S} \sum_{\substack{\partial m = xz \\ \partial n_S = \emptyset}} w(m)\, w(n_S)\, \mathbb{1}_{x \xleftrightarrow{m|_S + n_S} y}$$

switch.

$$= \sum_{y \in \partial_{in} S} \sum_{\substack{\partial m = yz \\ \partial n_S = xy}} w(m)\, w(n_S)$$

$$= \sum_{y \in \partial_{in} S} Z_S \langle xy \rangle_S\, Z_V \langle yz \rangle \qquad \blacksquare$$

4. GHS Inequality.

Not: For $A \subset V$ $\quad \langle A \rangle := \langle \sigma_A \rangle$

For $S \subset V$ $\quad n_S$: current with $(n_S)_e = 0 \ \forall e \notin S$.

$$Z_S[A] = \sum_{\partial n_S = A} w(n_S)$$

Thm [GHS ineq.]

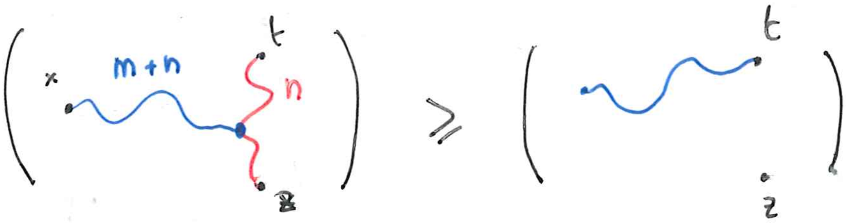
$x, y, z, t \in V$

$$\langle xyzt \rangle - \langle xy \rangle \langle zt \rangle - \langle xz \rangle \langle yt \rangle - \langle yz \rangle \langle xt \rangle + 2 \langle xt \rangle \langle yt \rangle \langle zt \rangle \leq 0 .$$

Lem:

$$\sum_{\substack{\partial m = xy \\ \partial n = zt}} w(m) w(n) 1_{x \longleftrightarrow t} \geq \langle zt \rangle \sum_{\substack{\partial m = xy \\ \partial n = \emptyset}} w(m) w(n) 1_{x \longleftrightarrow t}$$

Intuition



connecting x to z is easier if we already know that there exists a path from z to t in n.

Notation: $\mathcal{C}_x(n) = \{ v \in V : x \overset{n}{\longleftrightarrow} v \}$

Exercise: Let $C \subset V$ $\quad A, B \subset V$

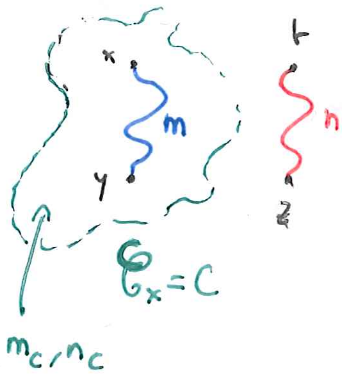
$$\sum_{\substack{\partial m = A \\ \partial n = B}} w(m) w(n) 1_{\mathcal{C}_x(m+n) = C} = \sum_{\substack{\partial m_C = A \cap C \\ \partial n_C = B \cap C}} w(m_C) w(n_C) 1_{\mathcal{C}_x(m_C + n_C) = C} \, Z_{V \setminus C}[A \setminus C] \, Z_{V \setminus C}[B \setminus C]$$

Proof of the Lemma:

$$\sum_{\substack{\partial m = xy \\ \partial n = zt}} w(m)\, w(n)\, 1\!\!1_{x \overset{m+n}{\longleftrightarrow} t} = Z[xy]\, Z[zt] - \sum_{\substack{\partial m = xy \\ \partial n = zt}} w(m)\, w(n)\, 1\!\!1_{x \overset{m+n}{\not\longleftrightarrow} t}$$

$$\langle zt \rangle \sum_{\substack{\partial m = xy \\ \partial n = \emptyset}} w(m)\, w(n)\, 1\!\!1_{x \overset{m+n}{\longleftrightarrow} t} = Z[xy]\, Z[zt] - \langle zt \rangle \sum_{\substack{\partial m = xy \\ \partial n = \emptyset}} w(m)\, w(n)\, 1\!\!1_{x \overset{m+n}{\not\longleftrightarrow} t}$$

$$\sum_{\substack{\partial m = xy \\ \partial n = zt}} w(m)\, w(n)\, 1\!\!1_{x \overset{m+n}{\not\longleftrightarrow} t} = \sum_{C :\, t \notin C} \ \sum_{\substack{\partial m = xy \\ \partial n = zt}} 1\!\!1_{\mathcal{C}_x(m+n) = C}\, w(m) w(n)$$



$$\overset{\text{ex}}{=} \sum_{C :\, t \notin C} \ \sum_{\substack{\partial m_C = xy \\ \partial n_C = \emptyset}} 1\!\!1_{\mathcal{C}_x(m_C + n_C) = C}\, w(m_C)\, w(n_C)\, Z_{V \setminus C} \cdot \underbrace{Z_{V\setminus C}[zt]}_{\substack{= Z_{V\setminus C} \cdot \langle zt \rangle_{V \setminus C} \\ (\text{GKS}) \\ \leq Z_{V\setminus C} \langle zt \rangle}}$$

$$\leq \langle zt \rangle \sum_{C :\, t \notin C} \ \sum_{\substack{\partial m = xy \\ \partial n = \emptyset}} 1\!\!1_{\mathcal{C}(m+n) = C}\, w(m)\, w(n)$$

$$= \langle zt \rangle \sum_{\substack{\partial m = xy \\ \partial n = \emptyset}} w(m)\, w(n)\, 1\!\!1_{x \overset{m+n}{\not\longleftrightarrow} t} \qquad \blacksquare$$

Proof of GHS inequality.

$$Z^2 \langle xy \rangle \langle zt \rangle = \sum_{\substack{\partial m = xy \\ \partial n = zt}} w(m)\, w(n) = \sum_{\substack{\partial m = xy\, zt \\ \partial n = \emptyset}} w(m)\, w(n)\, 1\!\!1_{z \overset{m+n}{\longleftrightarrow} t}.$$

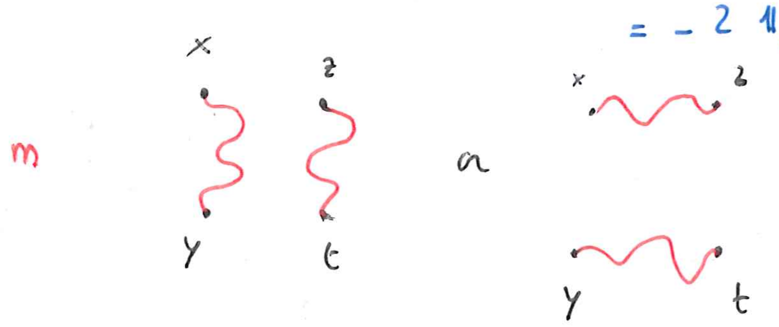
(Random current $Z = \sum_{\partial n = \emptyset} w(n)$)

$$Z^2\Big(\langle xyzt\rangle - \langle xy\rangle\langle zt\rangle - \langle xz\rangle\langle yt\rangle - \langle yz\rangle\langle xt\rangle\Big)$$

$$= \sum_{\substack{\partial m = xyzt \\ \partial n = \emptyset}} w(m)\, w(n) \Big(\underbrace{1 - \mathbb{1}_{x \xleftrightarrow{m+n} t} - \mathbb{1}_{y \xleftrightarrow{m+n} t} - \mathbb{1}_{z \xleftrightarrow{m+n} t}}\Big)$$

$$= -2\, \mathbb{1}_{xyzt \text{ all connected in } m+n}$$

$$= -2\, \mathbb{1}_{x \xleftrightarrow{m+n} t}\, \mathbb{1}_{z \xleftrightarrow{m+n} t}$$



switch.

$$= -2 \sum_{\substack{\partial m = xy \\ \partial n = zt}} \mathbb{1}_{x \xleftrightarrow{m+n} t}\; w(m)\, w(n)$$

lem

$$\leq -2 \langle zt\rangle \sum_{\substack{\partial m = xy \\ \partial n = \emptyset}} \mathbb{1}_{x \xleftrightarrow{m+n} t}\; w(m)\, w(n)$$

switch

$$= -2\langle zt\rangle \sum_{\substack{\partial m = yt \\ \partial n = xt}} w(m)\, w(n)$$

$$= Z^2\big(-2\langle zt\rangle\langle yt\rangle\langle xt\rangle\big) \qquad \blacksquare$$

CHAPTER 8 :

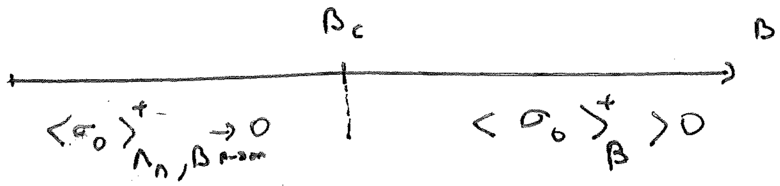
SHARP PHASE TRANSITION.

Ising in $\mathbb{Z}^d$, $d \geq 2$

n.n. interaction $\quad J_{xy} = \mathbb{1}_{x \sim y}$

no magnetic field $\quad h = 0$.

$$H^+_{\Lambda_n,\beta}(\sigma) = -\beta \sum_{xy \in E} \sigma_x \sigma_y \; - \beta \sum_{\substack{xy \in E \\ y \in \partial\Lambda_n}} \sigma_x$$



1 SHARPNESS

Thm [AIZENMAN - BARSKY - FERNANDEZ '87]

(i) $\forall \beta < \beta_c \quad \exists c > 0 \quad$ s.t.

$$\forall n \geq 1, \; \langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \leq e^{-cn}.$$

(ii) $\forall \beta \geq \beta_c$

$$\langle \sigma_0 \rangle^+_\beta \geq \frac{\sqrt{\beta - \beta_c}}{1 + \sqrt{\beta - \beta_c}}$$

Corollary:

(i) $\forall \beta < \beta_c \quad \exists c > 0 \quad$ s.t.

$$\forall x \quad \langle \sigma_0 \sigma_x \rangle_\beta \leq e^{-c\|x\|_\infty}$$ "exponential decay of correlation"

(ii) $\forall \beta \geq \beta_c$

$$\forall x \quad \langle \sigma_0 \sigma_x \rangle^+_\beta \geq \beta - \beta_c$$ "long-range order"

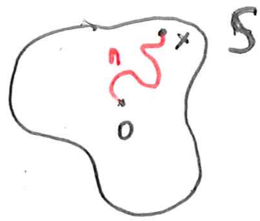
Pf: exercise

2. THE QUANTITY $\varphi_\beta(S)$

Def: Let $S \subset\subset \mathbb{Z}^d$

$$\varphi_\beta(S) := \begin{cases} \sum_{x \in \partial_{in} S} \langle \sigma_0 \sigma_x \rangle_S^{\emptyset} & \text{if } 0 \in S \\ 0 & \text{otherwise} \end{cases}$$

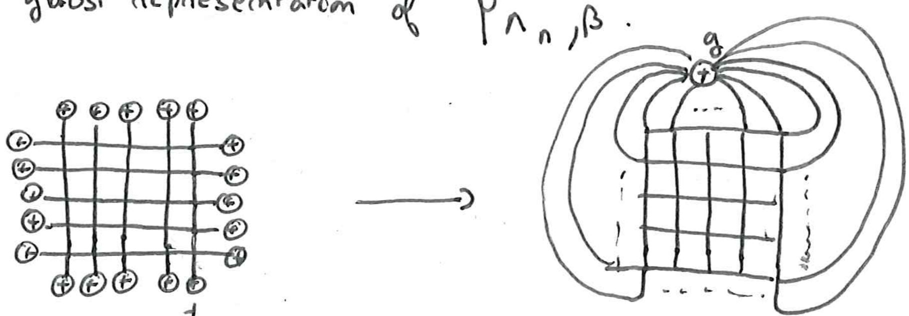
Current interpretation: $$\varphi_\beta(S) = \frac{1}{Z} \sum_{x \in \partial_{in} S} \underbrace{\sum_{\partial n = 0x} w(n)}$$



Lemma 1: Assume $\exists\, S \subset\subset \mathbb{Z}^d : 0 \in S$ $\varphi_\beta(S) < 1$.

Then $\exists\, c > 0$ s.t. $\forall n \geq 1$ $\langle \sigma_0 \rangle^+_{\Lambda_n, \beta} \leq e^{-cn}$.

Proof: "ghost representation of $\mu^+_{\Lambda_n,\beta}$".



Λ_n, $\mu^+_{\Lambda_n,\beta}$

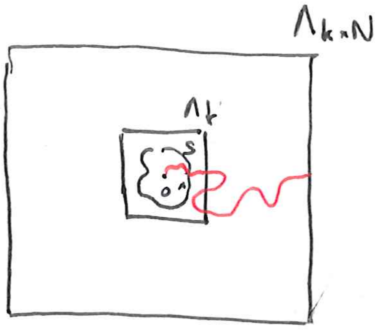
$G_n = V, E$ $\qquad V = \Lambda_n \cup \{g\}$

$J_{xy} = \beta\, \mathbb{1}_{x \sim y}$, $xy \in J_n$

$J_{xg} = \beta \times (\#\text{ edges from } x \text{ to } \Lambda_n^c)$

Let k s.t. $S \subset \Lambda_k$

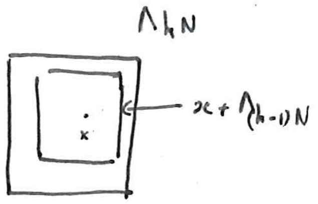
Let $n = k \times N$



$$\langle \sigma_0 \rangle^+_{\Lambda_{k \times N}, \beta} \underset{\uparrow}{=} \langle \sigma_0 \sigma_g \rangle_{G_n}$$

$$\overset{\text{Simon}}{\leq} \sum_{x \in \partial_{in} S} \langle \sigma_0 \sigma_x \rangle_S \underbrace{\langle \sigma_x \sigma_g \rangle_{G_n}}$$

$$= \langle \sigma_x \rangle^+_{\Lambda_{k \cdot N}, \beta}$$



$$\leq \langle \sigma_x \rangle^+_{x + \Lambda_{(k-1)N}, \beta}$$

$$= \langle \sigma_0 \rangle^+_{\Lambda_{(k-1)N}, \beta}.$$

$$\leq \phi_\beta(S) \langle \sigma_0 \rangle^+_{\Lambda_{(k-1)N}, \beta}.$$

By induction

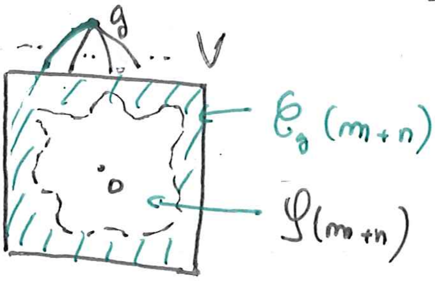
$$\langle \sigma_0 \rangle^+_{\Lambda_{kN}, \beta} \leq \phi_\beta(S)^k$$

∎

Lemma 2.

$$\frac{d}{d\beta}\langle\sigma_0\rangle^+_{\Lambda_n,\beta} \geq \frac{1}{Z^2}\sum_{\substack{\partial m=\emptyset\\ \partial n=\emptyset}} \phi_p\big(\mathcal{S}(m+n)\big)\, w(m)\,w(n).$$

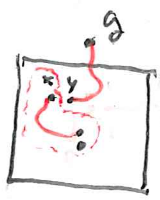
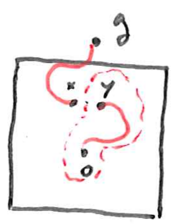
where $\mathcal{S}(m+n) = \{x\in V : x \overset{m+n}{\not\leftrightarrow} g\} = V\setminus \mathcal{E}_g(m+n)$



Proof: $$\frac{d}{d\beta}\langle\sigma_0\rangle^+_{\Lambda_n,\beta} = \sum_{xy\in E} \langle\sigma_0\sigma_x\sigma_y\rangle^+_{\Lambda_n,\beta} - \langle\sigma_0\rangle^+_{\Lambda_n,\beta}\langle\sigma_x\sigma_y\rangle^+_{\Lambda_n,\beta}$$

$$= \sum_{xy\in E} \langle 0xyg\rangle_V - \langle 0g\rangle_V\langle xy\rangle_V$$

$$= \frac{1}{Z^2}\sum_{xy\in E}\sum_{\substack{\partial m=0xyg\\ \partial n=\emptyset}} w(m)\,w(n)\,\mathbb{1}_{x\overset{m+n}{\not\leftrightarrow} y}.$$

$=$  $+$ 

$$\sum_{\substack{\partial m=\emptyset \\ \partial n=\emptyset}} \phi_p\left(\mathcal{C}_{m+n}\right) w(m)w(n) = \sum_{\substack{S \sqcup C = V \\ o \in S}} \sum_{\substack{\partial m=\emptyset \\ \partial n=\emptyset}} w(m)w(n)\, \mathbb{1}_{\mathcal{C}_g(m+n)=C}\, \phi_p(S)$$

" S, C partition of V "

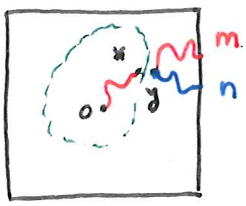
$$\underset{(ex)}{=} \sum_{\substack{S \sqcup C = V \\ o \in S}} \sum_{\substack{\partial m_C=\emptyset \\ \partial n_C=\emptyset}} w(m_C)w(n_C)\, \mathbb{1}_{\mathcal{C}_g(m_C+n_C)=C}\, Z_S \cdot \underbrace{Z_S\phi_p(S)}_{\leq \sum_{\substack{xy\in E \\ x\in S\, y\notin S}} Z_S[ox]}$$

$$\leq \sum_{xy\in E} \sum_{\substack{S \sqcup C = V \\ o,x\in S \\ y \in C}} \sum_{\substack{\partial m_C=\emptyset \\ \partial n_C=\emptyset}} w(m_C)w(n_C)\, \mathbb{1}_{\mathcal{C}_g(m_C+n_C)=C}\, Z_S \cdot Z_S[ox]$$

$$\underset{(ex)}{=} \sum_{xy\in E} \sum_{\substack{S \sqcup C = V \\ o,x\in S \\ y \in C}} \sum_{\substack{\partial m=ox \\ \partial n=\emptyset}} w(m)w(n)\, \mathbb{1}_{\mathcal{C}_g(m+n)=C}$$

$$= \sum_{xy\in E} \sum_{\substack{\partial m=ox \\ \partial n=\emptyset}} w(m)w(n)\, \mathbb{1}_{x \overset{m+n}{\not\leftrightarrow} y}\, \mathbb{1}_{g \overset{m+n}{\leftrightarrow} y}$$

$$= \sum_{xy\in E} \sum_{\substack{\partial m=oxgy \\ \partial n=og}} w(m)w(n)\, \mathbb{1}_{x \overset{m+n}{\not\leftrightarrow} y}$$



$$\underset{\substack{\uparrow \\ \text{as in lem} \\ \text{of previous section}}}{\leq} \sum_{xy\in E} \underbrace{\langle\sigma_y\rangle^+}_{\leq 1} \sum_{\substack{\partial m=oxyg \\ \partial n=\emptyset}} w(m)w(n)\, \mathbb{1}_{x \overset{m+n}{\not\leftrightarrow} y}$$

$$\leq Z^2 \frac{d}{d\beta} \langle\sigma_o\rangle^+_{\Lambda_n,\beta} \qquad \blacksquare$$

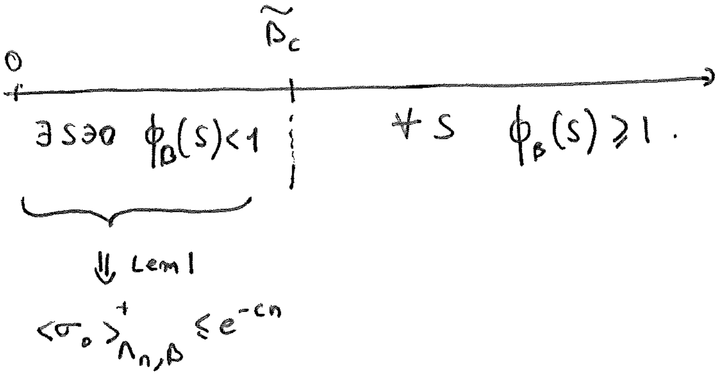
Rk: If in the last line we had $\langle \sigma_y \rangle^+ \simeq \langle \sigma_0 \rangle^+$

we would obtain $\frac{d}{d\beta}\left(\langle \sigma_0 \rangle^+_{\Lambda_n,\beta}\right)^2$ rather than $\frac{d}{d\beta}\left(\langle \sigma_0 \rangle_{\beta,\Lambda_n}\right)$

and this would conclude the theorem of Aizenman Barsky Fernandez with the correct mean field lower bound.

Proof of the theorem "bis" (with (ii) replaced by (ii') $\forall \beta \geq \beta_c \ \langle \sigma_0 \rangle_\beta \geq \frac{\beta - \beta_c}{1+\beta-\beta_c}$)

Let $\tilde{\beta}_c = \sup\{\beta : \exists S \subset\subset \mathbb{Z}^d, 0 \in S, \ \phi_\beta(S) < 1\}$



Fix $n \geq 1$, set $f(\beta) = \langle \sigma_0 \rangle^+_{\Lambda_n,\beta}$.

By lemma 2, we have $\forall \beta \geq \tilde{\beta}_c$

$$f' \geq \frac{1}{Z^2} \sum_{\substack{\partial m = \emptyset \\ \partial n = \emptyset}} w(m)\, w(n)\, \mathbb{1}_{0 \leftrightarrow g}$$

$$\overset{\text{switch}}{=} 1 - f^2 \geq 1 - f$$

Hence $\forall \beta \geq \tilde{\beta}_c$ $\quad \frac{f'}{1-f} \geq 1$

Integrating from $\tilde{\beta}_c$ to β we get $\log\left(\frac{1-f(\tilde{\beta}_c)}{1-f(\beta)}\right) \geq \beta - \tilde{\beta}_c$

$$\geq \log(1+\beta-\tilde{\beta}_c)$$

$x \geq \log(1+x)$

Therefore $f(\beta) \geq \frac{\beta - \tilde{\beta}_c}{1+(\beta-\tilde{\beta}_c)}$

CHAPTER 9

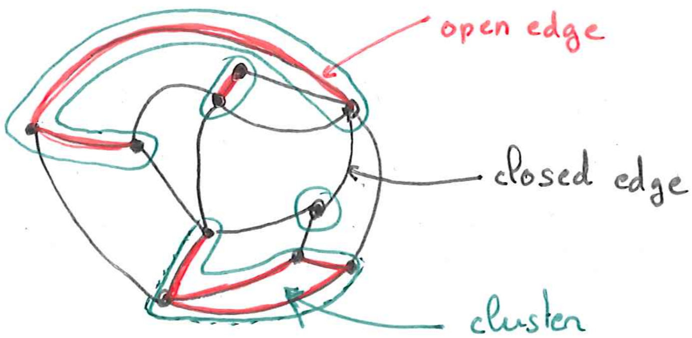
FK-PERCOLATION

$G = (V, E)$ finite graph $\quad p \in [0,1]$ "edge weight"

$q > 0$ "cluster weight".

I FK-PERCOLATION ON A FINITE GRAPH

Percolation configuration: $\omega = (\omega_e)_{e \in E} \in \{0,1\}^E$.



Rk: $\{\text{percolation config.}\} \overset{\text{bij}}{\simeq} \{\text{subgraphs of } G\}$

$\omega \longmapsto (V, \{e : \omega_e = 1\})$

Terminology:
- e is open in ω if $\omega_e = 1$
- e is closed in ω if $\omega_e = 0$
- cluster in ω = connected component of ω.
- open path in ω = path made of open edges.

Not:
- $|\omega| := \sum_{e \in E} \omega_e$ "number of open edges". (above $|\omega| = 7$)
- $|E \setminus \omega| = |E| - |\omega|$ "number of closed edges" (above $|E \setminus \omega| = 10$)
- $k(\omega)$ = "number of clusters in ω" ($k(\omega) = 4$)
- $A \overset{\omega}{\longleftrightarrow} B$: "$\exists$ open path in ω from A to B".

Def: The Fk-percolation measure on G with edge-weight p, cluster weight q is defined by

$$\boxed{\forall \omega \in \{0,1\}^E \quad \phi(\omega) = \frac{1}{Z} p^{|\omega|} (1-p)^{|E \setminus \omega|} q^{k(\omega)}}$$

where $Z = \sum_{\omega \in \{0,1\}^E} p^{|\omega|} (1-p)^{|E \setminus \omega|} q^{k(\omega)}$

Rk: • $q = 1 \to$ Bernoulli percolation $(\omega_e)_{e \in E}$ iid with $\omega_e \sim \text{Bernoulli}(p)$.

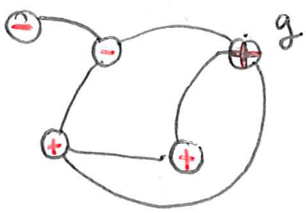
• As q increases, the measure ϕ_q favors config. with more disjoint clusters

• $\phi_{p,q} \xrightarrow[q \to 0]{} \begin{cases} \text{uniform connected subgraph} & p = \frac{1}{2} \\ \text{unif. spanning tree} & p \to 0 \quad \frac{q}{p} \to 0 \\ \text{unif. spanning forest} & p = q. \end{cases}$

2. EDWARDS – SOKAL COUPLING

$q = 2$

Fix $g \in V$ "ghost" $\quad \beta \geq 0 \quad \boxed{p = 1 - e^{-2\beta}}$



Ising on G: $\Omega^\circ = \{\sigma \in \{\pm 1\}^V : \sigma_g = +1\}$

$H^\circ(\sigma) = -\beta \sum_{xy \in E} \sigma_x \sigma_y \qquad \mu^\circ_\beta(\sigma) = \frac{1}{Z} e^{-H(\sigma)}$

Fk-perco $q = 2$

$$\phi(\omega) = \frac{1}{Z} p^{|\omega|} (1-p)^{|E \setminus \omega|} 2^{k(\omega)}$$

Not: $\sigma \sim \omega$: σ is constant on the clusters of ω
($x \overset{\omega}{\leftrightarrow} y \Rightarrow \sigma_x = \sigma_y$) "$\sigma$ is compatible with ω"

Prop: Let $\beta \geq 0$, $\boxed{p = 1 - e^{-2\beta}}$. The measure P on $\{0,1\}^E \times \Omega$ defined by

$$\boxed{\forall (\omega,\sigma) \quad P\big((\omega,\sigma)\big) = \frac{1}{Z}\, p^{|\omega|}(1-p)^{|E\setminus\omega|}\, \mathbb{1}_{\sigma\sim\omega}}$$

is a coupling of $\phi_{p,2}$ and μ°_β.
(ie $P(\{\omega\}\times\Omega^\circ) = \phi_{p,2}(\omega)$ and $P(\{0,1\}^E \times \{\sigma\}) = \mu^\circ(\sigma)$.

Proof:
$$\sum_{\sigma\in\Omega^\circ} P(\omega,\sigma) = \frac{1}{Z}\, p^{|\omega|}(1-p)^{|E\setminus\omega|} \times \underbrace{\sum_{\sigma} \mathbb{1}_{\sigma\sim\omega}}_{=2^{k(\omega)-1}}$$

(2 possibilities (+/-) for each cluster except the cluster of the ghost (+))

$$= \frac{1}{2Z}\, p^{|\omega|}(1-p)^{|E\setminus\omega|}\, 2^{k(\omega)}.$$

$$= \phi_{p,2}(\omega)$$

For $\sigma \in \Omega^\circ$, write $A_\sigma = \{xy \in E : \sigma_x = \sigma_y\}$ "agreement set"

$$\sum_{\omega\in\{0,1\}^E} P(\omega,\sigma) = \frac{(1-p)^{|E|}}{Z} \cdot \underbrace{\sum_{\omega\subset A_\sigma} \left(\frac{p}{1-p}\right)^{|\omega|}}_{=\prod_{e\in A_\sigma}\left(1+\frac{p}{1-p}\right)}$$

$$= \frac{e^{-2\beta|E|}}{Z} \times e^{2\beta|A_\sigma|}$$

$$= \frac{e^{-\beta|E|}}{Z} \times e^{H^\circ(\sigma)} = \mu^\circ(\sigma)$$

■

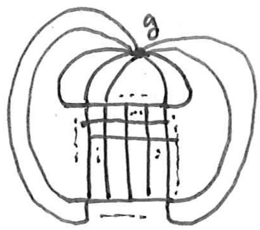
Important remark

$\forall (\eta, \xi) \in \{0,1\}^E \times \Omega^\circ$, we have

$$P[\sigma = \xi \mid \omega = \eta] = 1\!\!1_{\xi \sim \eta} \cdot \frac{1}{2^{k(\omega)-1}}$$

A random Ising configuration can be obtained by first sampling a $FK_{q=2}$ configuration and then coloring each cluster indep. $+1/-1$ with probabilities $\frac{1}{2}/\frac{1}{2}$, except the cluster of g which is colored $+1$.

Application

$G =$ 

$p = 1 - e^{-2\beta}$

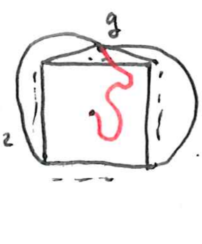
$(V = \{-n, \dots, n\}^d \cup \{g\})$

For $x \in V$

"coupling"

$$\langle \sigma_x \rangle_\beta^\circ = E^\circ[\sigma_x] = \underbrace{E[\sigma_x \mid x \overset{\omega}{\longleftrightarrow} g]}_{=+1} P[x \overset{\omega}{\longleftrightarrow} g] + \underbrace{E[\sigma_x \mid x \overset{\omega}{\not\longleftrightarrow} g]}_{=0} P[x \overset{\omega}{\not\longleftrightarrow} g]$$

$$= \phi_{p,2}(x \overset{\omega}{\longleftrightarrow} g)$$

$\langle \sigma_0 \rangle = \phi_{p,2}$ 

3 MONOTONICITY PROPERTIES

Lemma: For every configuration $\omega \in \{0,1\}^E$ $e = xy \in E$

$$\frac{\phi(\omega^e)}{\phi(\omega_e)} = \frac{p}{(1-p)\,q} \cdot q^{\mathbb{1}[x \overset{\omega_e}{\longleftrightarrow} y]}$$

Pf:

$$\frac{\phi(\omega^e)}{\phi(\omega_e)} = \frac{p}{1-p}\, q^{\overbrace{k(\omega^e) - k(\omega_e)}} \qquad = \begin{cases} 0 & \text{if } x \overset{\omega_e}{\longleftrightarrow} y \\ -1 & \text{if } x \overset{\omega_e}{\not\longleftrightarrow} y \end{cases}$$

Prop [FKG inequality]

Assume $q \geq 1$. Then $\forall A, B \uparrow$ events

$$\phi(A \cap B) \geq \phi(A)\,\phi(B)$$

Proof: Let $\eta \leq \psi$ $e = xy \in E$.

$$\frac{\phi(\eta^e)}{\phi(\eta_e)} \overset{\text{Lem}}{=} \frac{p}{(1-p)\,q} \cdot q^{\mathbb{1}_{x \overset{\eta_e}{\longleftrightarrow} y}} \overset{\boxed{q \geq 1}}{\leq} \frac{p}{(1-p)q}\, q^{\mathbb{1}_{x \overset{\psi_e}{\longleftrightarrow} y}} = \frac{\phi(\psi^e)}{\phi(\psi_e)}$$

Holley criterion applies.

Appli: For $q = 2$ $x, y, z \in V$

$$\langle \sigma_x \sigma_z \rangle = \phi(x \longleftrightarrow z) \overset{\text{FKG}}{\geq} \phi(x \longleftrightarrow y)\,\phi(y \longleftrightarrow z)$$

$$= \langle \sigma_x \sigma_y \rangle \langle \sigma_y \sigma_z \rangle.$$

→ We recover a particular case of GKS inequality.

Prop. $\boxed{\text{If } p \leq p' \text{ and } q \geq q' \geq 1, \text{ then } \phi_{p,q} \ll \phi_{p',q'}}$

Pf: $\eta \leq \psi$ $\quad e = xy \in E$

$$\frac{\phi_{p,q}(\eta^e)}{\phi_{p,q}(\eta_e)} \overset{\text{Lem}}{=} \frac{p}{(1-p)q}\, q^{\mathbb{1}[x \overset{\eta_e}{\longleftrightarrow} y]}$$

$$\overset{q\geq 1}{\leq} \frac{p}{(1-p)q}\, q^{\mathbb{1}[x \overset{\psi_e}{\longleftrightarrow} y]}$$

$$\leq \frac{p'}{(1-p')q'}\, q'^{\mathbb{1}[x \overset{\psi_e}{\longleftrightarrow} y]} = \frac{\phi(\psi^e)}{\phi(\psi_e)} \qquad \blacksquare$$

Appli: $q = 2$ $\qquad 0 \leq \beta \leq \beta'$ $\qquad x \in V$

$$\langle \sigma_x \rangle_\beta = \phi_{1-e^{-2\beta},2}(x \overset{w}{\longleftrightarrow} g) \leq \phi_{1-e^{-2\beta'},2}(x \overset{w}{\longleftrightarrow} g) = \langle \sigma_x \rangle_{\beta'}$$

↳ recover the monotonicity in β for Ising.

Prop: $\boxed{\text{If } 1 \leq q \leq q' \text{ and } \frac{p}{(1-p)q} \leq \frac{p'}{(1-p')q'}, \text{ then } \phi_{p,q} \ll \phi_{p',q'}}$

NB: If $1 \leq q \leq q'$ $\quad \phi_{p,q} \gg \phi_{p,q'}$ ← "same edge weight p", but if we increase "sufficiently" the edge weight p to $p' > p$, we get the stochastic domination in the other direction:

$$\phi_{p,q} \ll \phi_{p',q'}.$$

<u>Proof</u>:

$$\frac{\phi_{p,q}(\eta^e)}{\phi_{p,q}(\eta_e)} = \frac{p}{(1-p)q}\, q^{\mathbb{1}[x \overset{\eta_e}{\longleftrightarrow} y]} \leq \frac{p'}{(1-p')q'}\, q'^{\mathbb{1}[x \overset{\psi_e}{\longleftrightarrow} y]}$$

$$= \frac{\phi_{p',q'}(\psi^e)}{\phi_{p',q'}(\psi_e)} \qquad \blacksquare$$

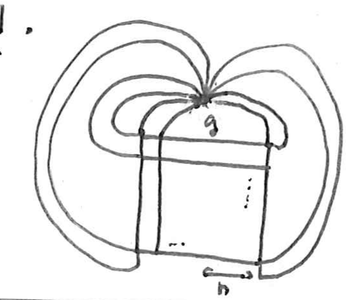
<u>Appli</u> $(q \geq 1)$

$$\underbrace{\phi_{p',1}}_{\text{Bernoulli}(p')} \ll \phi_{p,q} \ll \underbrace{\phi_{p,1}}_{\text{Bernoulli}(p)} \qquad \text{for } p' = \frac{p}{p+q(1-p)}$$

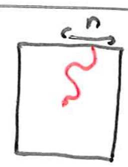
↪ useful to show that the phase transition for FK percolation is non trivial.

4 PHASE TRANSITION OF FK-PERCOLATION

<u>Fix $q \geq 1$.</u>

$G_n =$ 

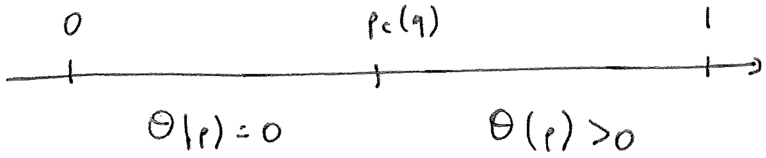
$V = \{-n, \dots, n\}^d \cup \{g\}$.

<u>Def</u>: $\boxed{\theta(p) = \lim_{n\to\infty} \phi_{G_n,p}(0 \longleftrightarrow g) = \lim_{n\to\infty} \phi_p\left[\ \right]}$ 

<u>Ex</u>: prove that θ is well defined.

Hint: first prove that $\forall k \quad \lim_{n\to\infty} \phi_{G_n,p}[0 \longleftrightarrow \partial\Lambda_k]$ exists

Def: $\boxed{p_c(q) = \sup\{p : \Theta(p) = 0\}}$.



Rk: for $q = 2$ $\qquad p_c(2) = 1 - e^{-2\beta_c}$

critical value for Ising.

Thm: [Beffara, Duminil-Copin '12]

$$\boxed{\text{for FK-percolation on } \mathbb{Z}^2 \text{, we have } \quad p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}}$$

Corollary:

$$\boxed{\text{For Ising on } \mathbb{Z}^2 \quad \beta_c = \frac{1}{2}\log\left(1+\sqrt{2}\right)}$$