ETH Zürich, Fall 2018
Prof. E. W. Farkas
Prof. M. Schweizer

Coordinator
Martin Stefanik

Mathematical Foundations for Finance

Exercise sheet 1

*This sheet contains material which is **fundamental** for this course and assumed to be known. Please hand in your solutions until Tuesday, 25/09/2018, 18:00 into your assistant's box next to HG G 53.2.*

Exercise 1.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with $\Omega := \{UU, UD, DD, DU\}$, $\mathcal{F} := 2^\Omega$ and P defined by $P[\omega] := 1/4$ for all $\omega \in \Omega$. Let $Y_1, Y_2 : \Omega \to \mathbb{R}$ be two random variables with $Y_1(UU) = Y_1(UD) := 2$, $Y_1(DD) = Y_1(DU) := 1/2$, $Y_2(UU) = Y_2(DU) := 2$ and $Y_2(DD) = Y_2(UD) := 1/2$. Define the process $X = (X_k)_{k=0,1,2}$ by

$$X_0(\omega) = 8 \quad \text{for all } \omega \in \Omega,$$
$$X_k(\omega) = X_0(\omega) \prod_{i=1}^{k} Y_i(\omega) \quad \text{for } k = 1, 2.$$

(a) Explicitly write down the sequences of σ-fields $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ and $\mathbb{G} = (\mathcal{G}_k)_{k=0,1,2}$ defined by $\mathcal{F}_k := \sigma(X_i, 0 \le i \le k)$ and $\mathcal{G}_k := \sigma(X_k)$, $k = 0, 1, 2$.

(b) Show that $Z : \Omega \to \mathbb{R}$ defined by $Z(\omega) := 2X_1(\omega) + 1$ is $\sigma(X_1)$-measurable.

(c) Do $\mathbb{F}$ and $\mathbb{G}$ form filtrations on $(\Omega, \mathcal{F})$? Why or why not?

(d) Is X adapted to $\mathbb{F}$ or $\mathbb{G}$ (in case any of the former is a filtration on $(\Omega, \mathcal{F})$)?

(e) Try to give financial interpretations for X and $\mathbb{F}$.

Exercise 1.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ a random variable with $X \ge 0$ P-a.s. Prove that $E[X] = 0$ implies that $X = 0$ P-a.s.
Hint: Find a way to use the monotone convergence theorem.

Exercise 1.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space, X an integrable random variable and $\mathcal{G} \subseteq \mathcal{F}$ a σ-field. Then the P-a.s. unique random variable Z such that

- Z is $\mathcal{G}$-measurable and integrable,
- $E[X\mathbb{1}_A] = E[Z\mathbb{1}_A]$ for all $A \in \mathcal{G}$,

is called *conditional expectation of X given $\mathcal{G}$* and is denoted by $E[X \,|\, \mathcal{G}]$. (This is the formal definition of conditional expectation of X given $\mathcal{G}$; see Section 8.2 in the lecture notes.)

(a) Use the definition above to show that if X is $\mathcal{G}$-measurable, then $E[X \,|\, \mathcal{G}] = X$ P-a.s.

(b) Use the definition of conditional expectation to show that $E[E[X \,|\, \mathcal{G}]] = E[X]$.

(c) Use the definition of conditional expectation to show that if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{G}$, i.e. if $\mathcal{G}$ is P-trivial, then $E[X \,|\, \mathcal{G}] = E[X]$ P-a.s.

(d) Consider an integrable random variable Y on $(\Omega, \mathcal{F}, P)$, and two constants $a, b \in \mathbb{R}$. Use the definition of conditional expectation to show that $E[aX + bY \,|\, \mathcal{G}] = aE[X \,|\, \mathcal{G}] + bE[Y \,|\, \mathcal{G}]$ P-a.s.

(e) Suppose that $\mathcal{G}$ is generated by a finite partition of Ω, i.e. there exists a collection $(A_i)_{i=1}^n$ of sets $A_i \in \mathcal{F}$ such that $\bigcup_{i=1}^n A_i = \Omega$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\mathcal{G} = \sigma(A_1, \dots, A_n)$. Additionally, assume that $P[A_i] > 0$ for all $i = 1, \dots, n$. Use the definition of conditional expectation to show that

$$E\left[X \mid \mathcal{G}\right] = \sum_{i=1}^{n} E\left[X \mid A_i\right] \mathbb{1}_{A_i} \qquad P\text{-a.s.}$$

Hint 1: Recall that $E\left[X \mid A_i\right] = E\left[X \mathbb{1}_{A_i}\right] / P[A_i]$ *and try to write* X *as a sum of random variables each of which only takes non-zero values on a single* A_i*.*
Hint 2: Check that any set $A \in \mathcal{G}$ *is of the form* $\cup_{j \in J} A_j$ *for some* $J \subseteq \{1, \dots, n\}$*.*