Mathematical Foundations for Finance

Exercise sheet 1

Exercise 1.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with $\Omega := \{UU, UD, DD, DU\}$, $\mathcal{F} := 2^\Omega$ and P defined by $P[\omega] := 1/4$ for all $\omega \in \Omega$. Let $Y_1, Y_2 : \Omega \to \mathbb{R}$ be two random variables with $Y_1(UU) = Y_1(UD) := 2$, $Y_1(DD) = Y_1(DU) := 1/2$, $Y_2(UU) = Y_2(DU) := 2$ and $Y_2(DD) = Y_2(UD) := 1/2$. Define the process $X = (X_k)_{k=0,1,2}$ by

$$X_0(\omega) = 8 \quad \text{for all } \omega \in \Omega,$$
$$X_k(\omega) = X_0(\omega) \prod_{i=1}^{k} Y_i(\omega) \quad \text{for } k = 1, 2.$$

(a) Explicitly write down the sequences of σ-fields $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ and $\mathbb{G} = (\mathcal{G}_k)_{k=0,1,2}$ defined by $\mathcal{F}_k := \sigma(X_i, 0 \leq i \leq k)$ and $\mathcal{G}_k := \sigma(X_k)$, $k = 0, 1, 2$.

(b) Show that $Z : \Omega \to \mathbb{R}$ defined by $Z(\omega) := 2X_1(\omega) + 1$ is $\sigma(X_1)$-measurable.

(c) Do $\mathbb{F}$ and $\mathbb{G}$ form filtrations on $(\Omega, \mathcal{F})$? Why or why not?

(d) Is X adapted to $\mathbb{F}$ or $\mathbb{G}$ (in case any of the former is a filtration on $(\Omega, \mathcal{F})$)?

(e) Try to give financial interpretations for X and $\mathbb{F}$.

Solution 1.1

(a) We have that $\mathcal{F}_0 = \mathcal{G}_0$ by definition with $\mathcal{F}_0 = \{\{X_0 = 8\}, \{X_0 \neq 8\}\} = \{\emptyset, \Omega\}$. Since $\mathcal{F}_0 = \mathcal{G}_0$ are contained in any other σ-field (by the definition of σ-field), we also have that $\mathcal{F}_1 = \sigma(X_0, X_1) = \sigma(X_1) = \mathcal{G}_1$. Furthermore,

$$\begin{aligned}
\mathcal{G}_1 = \sigma(X_1) &= \big\{\{X_1 = 16\}, \{X_1 = 4\}, \{X_1 = 16\} \cup \{X_1 = 4\}, \{X_1 \neq 16, X_1 \neq 4\}\big\} \\
&= \big\{\{Y_1 = 2\}, \{Y_1 = 1/2\}, \{Y_1 = 2\} \cup \{Y_1 = 1/2\}, \{Y_1 \neq 2, Y_1 \neq 1/2\}\big\} \\
&= \big\{\{Y_1 = 2\}, \{Y_1 = 1/2\}, \Omega, \emptyset\big\} \\
&= \big\{\{UU, UD\}, \{DD, DU\}, \Omega, \emptyset\big\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}_2 = \sigma(X_2) &= \big\{\{X_2 = 32\}, \{X_2 = 8\}, \{X_2 = 2\}, \{X_2 = 32\} \cup \{X_2 = 8\}, \\
&\qquad \{X_2 = 32\} \cup \{X_2 = 2\}, \{X_2 = 8\} \cup \{X_2 = 2\}, \Omega, \emptyset\big\} \\
&= \big\{\{Y_1 = 2, Y_2 = 2\}, \{Y_1 = 1/2, Y_2 = 2\} \cup \{Y_1 = 2, Y_2 = 1/2\}, \\
&\qquad \{Y_1 = 1/2, Y_2 = 1/2\}, \\
&\qquad \{Y_1 = 2, Y_2 = 2\} \cup \{Y_1 = 1/2, Y_2 = 2\} \cup \{Y_1 = 2, Y_2 = 1/2\}, \\
&\qquad \{Y_1 = 2, Y_2 = 2\} \cup \{Y_1 = 1/2, Y_2 = 1/2\}, \\
&\qquad \{Y_1 = 1/2, Y_2 = 2\} \cup \{Y_1 = 2, Y_2 = 1/2\} \cup \{Y_1 = 1/2, Y_2 = 1/2\}, \Omega, \emptyset\big\}, \\
&= \big\{\{UU\}, \{DU, UD\}, \{DD\}, \{UU, DU, UD\}, \{UU, DD\}, \{DU, UD, DD\}, \\
&\qquad \Omega, \emptyset\big\},
\end{aligned}$$

$$\mathcal{F}_2 = \sigma(X_1, X_2) = \quad \mathcal{F}.$$

(b) Since $g(x) = 2x + 1$ is a continuous function, we immediately get that $Z = g(X_1)$ is $\sigma(X_1)$-measurable. One could also argue by writing out $\sigma(Z)$ explicitly and showing that $\sigma(Z) \subseteq \sigma(X_1)$. We have that

$$\begin{aligned}\sigma(Z) &= \big\{\{Z = 33\}, \{Z = 9\}, \{Z = 33\} \cup \{Z = 9\}, \{Z \neq 33\, Z \neq 9\}\big\} \\ &= \big\{\{Z = 33\}, \{Z = 9\}, \Omega, \emptyset\big\} \\ &= \big\{\{UU, UD\}, \{DD, DU\}, \Omega, \emptyset\big\} = \sigma(X_1).\end{aligned}$$

Yet another approach would be to show that all sets of the form $\{Z \le c\}$ for $c \in \mathbb{R}$ lie in $\mathcal{F}$. This works because X_1 is measurable with respect to $\mathcal{F}_1 \subseteq \mathcal{F}$. However,

$$\{Z \le c\} = \{2X_1 + 1 \le c\} = \left\{X_1 \le \frac{c-1}{2}\right\} \in \mathcal{F},$$

since X_1 is also $\mathcal{F}$-measurable.

(c) We know that a filtration $\mathbb{H} = (\mathcal{H}_k)_{k=1,\dots,T}$, $T \in \mathbb{N}$ on a measurable space $(\Omega, \mathcal{F})$ is an increasing family of σ-fields $\mathcal{H}_k \subseteq \mathcal{F}$ in the sense that $\mathcal{H}_k \subseteq \mathcal{H}_n$ for $k \le n$. $\mathbb{F}$ clearly forms a filtration because $\mathcal{F}_0 = \{\emptyset, \Omega\} \subseteq \mathcal{F}_1$ since $\mathcal{F}_0$ is contained in any σ-field and $\mathcal{F}_1 \subseteq \mathcal{F}_2$ because $\sigma(X_1) \subseteq \sigma(X_1, X_2)$. On the other hand, $\mathbb{G}$ does not form a filtration since $\{DD, DU\} \in \mathcal{G}_1$ but at the same time $\{DD, DU\} \notin \mathcal{G}_2$. Therefore $\mathcal{G}_1 \nsubseteq \mathcal{G}_2$.

(d) Since $\mathbb{G}$ does not form a filtration, we are only interested in whether X is adapted to $\mathbb{F}$. In order to decide this, we need to check that X_k is $\mathcal{F}_k$-measurable for $k = 0, 1, 2$. This is, however, trivial since we have constructed $\mathcal{F}_1$ (by definition of σ-field generated by a random variable) as the smallest σ-field such that X_1 is $\mathcal{F}_1$-measurable, and $\mathcal{F}_2$ as the smallest σ-field such that X_1 and X_2 are $\mathcal{F}_2$-measurable. For this reason, $\sigma(X_i, 1 \le i \le k)$ is also called the *canonical filtration* for X_k.

(e) The process X can be interpreted as the price of a stock that is only allowed to move in up or down by factors of 2 and 1/2 respectively in each period.

The filtration $\mathbb{F}$ can be thought of as the *cumulative* information that the stock price evolution provides us with over time.

Exercise 1.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ a random variable with $X \ge 0$ P-a.s. Prove that $E[X] = 0$ implies that $X = 0$ P-a.s.
Hint: Find a way to use the monotone convergence theorem.

Solution 1.2 Since $X \ge 0$ P-a.s., it is sufficient to show that $P[X > 0] = 0$. Markov's inequality together with our assumption that $E[X] = 0$ imply that

$$P\left[X \ge \frac{1}{n}\right] \le nE[X] = 0 \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

But since

$$\left\{X \ge \frac{1}{n}\right\} \subseteq \left\{X \ge \frac{1}{n+1}\right\} \quad \text{for all } n \in \mathbb{N}, \tag{2}$$

we obtain that

$$\mathbb{1}_{\{X \ge \frac{1}{n}\}} \le \mathbb{1}_{\{X \ge \frac{1}{n+1}\}} \quad P\text{-a.s. for all } n \in \mathbb{N}.$$

So $\mathbb{1}_{\{X \ge \frac{1}{n}\}}$, $n \in \mathbb{N}$ form a sequence of almost surely increasing functions and therefore

$$\begin{aligned}\lim_{n\to\infty} P\left[X \ge \frac{1}{n}\right] &= \lim_{n\to\infty} E\left[\mathbb{1}_{\{X \ge \frac{1}{n}\}}\right] = E\left[\lim_{n\to\infty} \mathbb{1}_{\{X \ge \frac{1}{n}\}}\right] = E\left[\lim_{n\to\infty} \mathbb{1}_{\{\cup_{k=1}^n \{X \ge \frac{1}{n}\}\}}\right] \\ &= E\left[\mathbb{1}_{\{\lim_{n\to\infty} \cup_{k=1}^n \{X \ge \frac{1}{n}\}\}}\right] = E\left[\mathbb{1}_{\{X>0\}}\right] = P[X > 0],\end{aligned}$$

where the second equality follows from the monotone convergence theorem and the third equality follows from (2). Combining the above with (1), we conclude that

$$P[X > 0] = \lim_{n\to\infty} P\left[X \geq \frac{1}{n}\right] = 0.$$

Exercise 1.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space, X an integrable random variable and $\mathcal{G} \subseteq \mathcal{F}$ a σ-field. Then the P-a.s. unique random variable Z such that

- Z is $\mathcal{G}$-measurable and integrable,
- $E[X\mathbb{1}_A] = E[Z\mathbb{1}_A]$ for all $A \in \mathcal{G}$,

is called *conditional expectation of X given $\mathcal{G}$* and is denoted by $E[X \,|\, \mathcal{G}]$. (This is the formal definition of conditional expectation of X given $\mathcal{G}$; see Section 8.2 in the lecture notes.)

(a) Use the definition above to show that if X is $\mathcal{G}$-measurable, then $E[X \,|\, \mathcal{G}] = X$ P-a.s.

(b) Use the definition of conditional expectation to show that $E[E[X \,|\, \mathcal{G}]] = E[X]$.

(c) Use the definition of conditional expectation to show that if $P[A] \in \{0, 1\}$ for all $A \in \mathcal{G}$, i.e. if $\mathcal{G}$ is P-trivial, then $E[X \,|\, \mathcal{G}] = E[X]$ P-a.s.

(d) Consider an integrable random variable Y on $(\Omega, \mathcal{F}, P)$, and two constants $a, b \in \mathbb{R}$. Use the definition of conditional expectation to show that $E[aX + bY \,|\, \mathcal{G}] = aE[X \,|\, \mathcal{G}] + bE[Y \,|\, \mathcal{G}]$ P-a.s.

(e) Suppose that $\mathcal{G}$ is generated by a finite partition of Ω, i.e. there exists a collection $(A_i)_{i=1}^n$ of sets $A_i \in \mathcal{F}$ such that $\bigcup_{i=1}^n A_i = \Omega$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\mathcal{G} = \sigma(A_1, \dots, A_n)$. Additionally, assume that $P[A_i] > 0$ for all $i = 1, \dots, n$. Use the definition of conditional expectation to show that

$$E[X \,|\, \mathcal{G}] = \sum_{i=1}^n E[X \,|\, A_i] \mathbb{1}_{A_i} \qquad P\text{-a.s.}$$

Hint 1: Recall that $E[X \,|\, A_i] = E[X\mathbb{1}_{A_i}]/P[A_i]$ and try to write X as a sum of random variables each of which only takes non-zero values on a single A_i.
Hint 2: Check that any set $A \in \mathcal{G}$ is of the form $\cup_{j\in J} A_j$ for some $J \subseteq \{1, \dots, n\}$.

Solution 1.3

(a) X is $\mathcal{G}$-measurable and integrable by assumption, so the first requirement in the definition of conditional expectation is satisfied for $Z = X$. Moreover, we clearly have that $E[X\mathbb{1}_A] = E[X\mathbb{1}_A]$ for all $A \in \mathcal{G}$, hence $E[X \,|\, \mathcal{G}] = X$ P-a.s.

(b) In the definition of conditional expectation set $A = \Omega$. We have that $E[E[X \,|\, \mathcal{G}]] = E[E[X \,|\, \mathcal{G}]\mathbb{1}_\Omega] = E[X\mathbb{1}_\Omega] = E[X]$.

(c) Since $|E[X]| \leq E[|X|]$ by Jensen's inequality and $E[|X|] < \infty$ since X is integrable by assumption, we have that $E[X]$ is integrable as well. $E[X]$ is also trivially $\mathcal{G}$-measurable since it is a constant random variable. Moreover, in this setting, $A \in \mathcal{G}$ only if $P[A] = 0$ or $P[A] = 1$. Noting that

$$\begin{aligned} E[X\mathbb{1}_A] &= 0 = E[E[X]\mathbb{1}_A] && \forall\, A \in \mathcal{G} \text{ such that } P[A] = 0, \\ E[X\mathbb{1}_A] &= E[X] = E[E[X]\mathbb{1}_A] && \forall\, A \in \mathcal{G} \text{ such that } P[A] = 1, \end{aligned}$$

we obtain $E[X \,|\, \mathcal{G}] = E[X]$ P-a.s.

(d) By the definition of conditional expectation, we have that $E[X \mid \mathcal{G}]$ and $E[Y \mid \mathcal{G}]$ are $\mathcal{G}$-measurable and integrable, hence the same holds for $aE[X \mid \mathcal{G}] + bE[Y \mid \mathcal{G}]$. Choosing some $A \in \mathcal{G}$ we can compute that

$$\begin{aligned} E\left[\left(aE[X \mid \mathcal{G}] + bE[Y \mid \mathcal{G}]\right)\mathbb{1}_A\right] &= aE\left[E[X \mid \mathcal{G}]\,\mathbb{1}_A\right] + bE\left[E[Y \mid \mathcal{G}]\,\mathbb{1}_A\right] \\ &= aE[X\mathbb{1}_A] + bE[Y\mathbb{1}_A] = E\left[\left(aX + bY\right)\mathbb{1}_A\right], \end{aligned}$$

where the first equality uses the linearity of (classical) expectation and the second uses the definition of $E[X \mid \mathcal{G}]$ and $E[Y \mid \mathcal{G}]$. This shows that $E[aX + bY \mid \mathcal{G}] = aE[X \mid \mathcal{G}] + bE[Y \mid \mathcal{G}]$ P-a.s.

(e) First recall that $E[X \mid A_i] = E[X\mathbb{1}_{A_i}]/P[A_i]$. Using that $X = \sum_{i=1}^n X\mathbb{1}_{A_i}$, we get by (d) that

$$E[X \mid \mathcal{G}] = \sum_{i=1}^n E[X\mathbb{1}_{A_i} \mid \mathcal{G}] \quad P\text{-a.s.},$$

and hence we only have to show that $E[X\mathbb{1}_{A_i} \mid \mathcal{G}] = \frac{E[X\mathbb{1}_{A_i}]}{P[A_i]}\mathbb{1}_{A_i}$ P-a.s. for each $i \in \{1, \dots, n\}$. Since $A_i \in \mathcal{G}$ and $E[X \mid A_i] = E[X\mathbb{1}_{A_i}]/P[A_i] \in \mathbb{R}$, we already know that $E[X \mid A_i]\,\mathbb{1}_{A_i}$ is $\mathcal{G}$-measurable and integrable. One can verify that the family of sets $A = \bigcup_{j \in J} A_j$ for $J \in 2^{\{1,\dots,n\}}$ (the power set of $\{1, \dots, n\}$) forms a σ-field. Let's denote this σ-field by $\tilde{\mathcal{G}}$. Since we clearly have that $A_i \in \tilde{\mathcal{G}}$ for all $i \in \{1, \dots, n\}$, we get that $\tilde{\mathcal{G}} \supseteq \mathcal{G}$, which for any $A \in \mathcal{G}$ implies that $A = \bigcup_{j \in J} A_j$ for some $J \subseteq \{1, \dots, n\}$. For any such $A \in \mathcal{G}$ we have that

$$\mathbb{1}_{A_i}\mathbb{1}_A = \begin{cases} \mathbb{1}_{A_i} & \text{if } i \in J, \\ 0, & \text{else.} \end{cases}$$

Hence we can then compute

$$E\left[\left(\frac{E[X\mathbb{1}_{A_i}]}{P[A_i]}\mathbb{1}_{A_i}\right)\mathbb{1}_A\right] = \begin{cases} E[X\mathbb{1}_{A_i}]\,\frac{P[A_i]}{P[A_i]} = E[X\mathbb{1}_{A_i}] & \text{if } i \in J, \\ 0 & \text{else.} \end{cases}$$

On the other hand, we have that

$$E[X\mathbb{1}_{A_i}\mathbb{1}_A] = \begin{cases} E[X\mathbb{1}_{A_i}] & \text{if } i \in J, \\ 0 & \text{else.} \end{cases}$$

This shows that $E[X\mathbb{1}_{A_i} \mid \mathcal{G}] = \frac{E[X\mathbb{1}_{A_i}]}{P[A_i]}\mathbb{1}_{A_i}$ P-a.s. and concludes the proof.