ETH zürich

Mathematical Foundations for Finance

Exercise 10

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Quadratic Variation & Covariation

Theorem 1

For any local martingale $M = (M_t)_{t\geq 0}$ null at 0, there exists a unique adapted increasing RCLL process $[M] = ([M]_t)_{t\geq 0}$ null at zero with $\Delta[M] = (\Delta M)^2$ having the property that $M^2 - [M]$ is a local martingale. There exists a sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ of $[0,\infty)$ with $|\Pi_n| \to 0$ as $n \to \infty$ such that

$$P\left[[M]_t(\omega) = \lim_{n\to\infty} \sum_{t_i\in\Pi_n} \left(M_{t_i\wedge t}(\omega) - M_{t_{i-1}\wedge t}(\omega)\right)^2 \text{ for all } t \geq 0\right] = 1.$$

Definition 2 (Covariation process)

Let M and N be two local martingales that are both null at 0. We define the *covariation process* of M and N by

$$[M,N] = \frac{1}{4}\left([M+N] - [M-N]\right).$$

From the previous definition of quadratic covariation, is not be surprising that we can also express $[M, N]$ as a limit of a similar expression. Indeed, there exists a sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ of $[0,\infty)$ with $|\Pi_n| \to 0$ as $n \to \infty$ such that

$$P\left[[M,N]_t(\omega) = \lim_{n\to\infty} \sum_{t_i\in\Pi_n} \left(M_{t_i\wedge t}(\omega) - M_{t_{i-1}\wedge t}(\omega)\right)\left(N_{t_i\wedge t}(\omega) - N_{t_{i-1}\wedge t}(\omega)\right) \text{ for all } t \geq 0\right] = 1.$$

Quadratic Variation & Covariation of Local Martingales

What can we say about the properties of $[\cdot]$ and $[\cdot,\cdot]$ from the above?

- $[aM] = a^2[M]$ P-a.s., where $a \in \mathbb{R}$ is a constant.
- Reverse expression for $[M]$ using $[\cdot,\cdot]$

$$[M, M] = \frac{1}{4}\left([2M] - [M - M]\right) = [M].$$

- Symmetry of $[\cdot,\cdot]$

$$[M, N] = \frac{1}{4}\left([M + N] - [M - N]\right) = \frac{1}{4}\left([N + M] - [N - M]\right) = [N, M].$$

- Bilinearity of $[\cdot,\cdot]$

$$[aL + M, N] = [aL, N] + [M, N] = a[L, N] + [M, N],$$
$$[L, aM + N] = [L, aM] + [L, N] = a[L, M] + [L, N],$$

  for a constant $a \in \mathbb{R}$.

- Bilinearity and symmetry also imply that $[M + N] = [M] + 2[M, N] + [N]$.

Quadratic Variation & Covariation of Semimartingales

We extend the definition of quadratic variation to a semimartingale $X = X_0 + M + A$ by defining

$$[X] := [M] + 2\sum \Delta M \Delta A + \sum (\Delta A)^2$$

In this case we have

- When $X_0 = A = 0$, the above reduces previous case, i.e. $[X] = [M]$ (consistency).
- X_0 is random variable, not a process, so it does not contribute to $[X]$.
- When $M = 0$ and $X = A$ is continuous, then we have that $[X] = 0$. This is in line with $[A] = 0$, which we have seen already (e.g. Exercise 8.2)
- We can define $[X, Y]$ for two semimartingales X and Y as before by polarization.
- $[\cdot]$ and $[\cdot, \cdot]$ then keep all the properties from the previous slide.

Quadratic Variation & Covariation of Semimartingales

Let X, Y, Z be general general semimartingales and $a \in \mathbb{R}$ a constant. Note again that

1. $[X, Y] = [Y, X]$
2. $[aX, Y] = a[X, Y]$
3. $[X + Z, Y] = [X, Y] + [Z, Y]$
4. $[X, X] = [X] \geq 0$
5. $X = 0 \implies [X, X] = 0$

This means that $[\cdot, \cdot]$ almost defines an inner product on the space of semimartingales, except that we do not have that $X = 0 \iff [X, X] = 0$. Nevertheless, we still have Cauchy-Schwarz, i.e.

$$|[X, Y]_t|^2 \leq [X]_t [Y]_t.$$

Since all local martingales and all finite variation processes are by definition semimartingales, we can apply the above to an arbitrary local martingale M and a continuous finite variation process A and we obtain that $[M, A]_t = 0$.

Itô's Formula for Continuous Semimartingales

Theorem 3 (Itô's formula I)

Suppose $X = (X_t)_{t\geq 0}$ is a continuous real-valued semimartingale and $f : \mathbb{R} \to \mathbb{R}$ is in C^2. Then $f(X) = (f(X_t))_{t\geq 0}$ is again a continuous (real-valued) semimartingale, and for all $t \geq 0$ we explicitly have P-a.s. that

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X\rangle_s.$$

- Written in a more compact *formal* notation as

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t.$$

- Be careful not to forget about $f(X_0)$ when working in the above differential form.
- The above formula also shows another reason why the properties of the quadratic variation are of interest.

Thank you for your attention!