Mathematical Foundations for Finance

Exercise sheet 10

Exercise 10.1 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions. Define

$$\tau_a := \inf\{t \geq 0 \mid W_t > a\}$$

for some $a > 0$.

(a) Prove that τ_a is a stopping time for all $a > 0$, and that we have $\tau_{a_1} \leq \tau_{a_2}$ P-a.s. for $a_1 < a_2$.
Hint 1: Use that fact that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then if $f(x) > a$ for some $x, a \in \mathbb{R}$, there exists a $y \in \mathbb{Q}$ arbitrarily close to x such that $f(y) > a$.
Hint 2: Use that the filtration is right-continuous, i.e. if $A \in \mathcal{F}_{t+1/n}$ for all $n \in \mathbb{N}$, then $A \in \mathcal{F}_t$.

(b) Prove that $P[\tau_a < \infty] = 1$ for all $a > 0$.
Hint: Use the global of the iterated logarithm from Proposition V.1.2 in the lecture notes.

(c) Show that $W_{\tau_a} = a$ P-a.s. for all $a > 0$ and conclude that

$$E\left[W_{\tau_{a_2}} \,\middle|\, \mathcal{F}_{\tau_{a_1}}\right] \neq W_{\tau_{a_1}} \quad P\text{-a.s.},$$

for $a_1 < a_2$, proving that the stopping theorem (Theorem IV.2.1 in the lecture notes) fails for $\tau = \tau_{a_2}$ and $\sigma = \tau_{a_1}$.

(d) Prove that $\rho_a := \sup\{t \geq 0 | W_t > a\}$ is a stopping time. What values does it take?
Hint: Use that the filtration is P-complete, i.e. if $P[A] = 0$ for some $A \in \mathcal{F}$, then $A \in \mathcal{F}_0$.

Solution 10.1

(a) Fix an $a > 0$ and a $t > 0$. We have to show that $\{\tau_a \leq t\} \in \mathcal{F}_t$. Observe that

$$\{\tau_a < t\} = \{W_s > a \text{ for some } s \in [0, t)\} = \{W_s > a \text{ for some } s \in [0, t]\}$$

because W has P-a.s. continuous trajectories. Note that we cannot have that

$$\{\tau_a \leq t\} = \{W_s > a \text{ for some } s \in [0, t]\}$$

because for a fixed $\omega \in \Omega$ it can happen that $\tau_a = t$ even if $W_s(\omega) \not> a$ for any $s \in [0, t]$. This is the case when $W_t(\omega) = a$ and $W_{t+\epsilon}(\omega) > a$ for all $\epsilon \in (0, \delta)$ for some $\delta > 0$. This also suggests why the right continuity of the filtration is needed.

Now, it is clear that we can write

$$\{\tau_a < t\} = \{W_s > a \text{ for some } s \in [0, t]\} = \bigcup_{s\in[0,t]} \{W_s > a\}.$$

However, since the set $[0, t]$ is an uncountable one, the union is uncountable as well, and since a σ-algebra is by definition closed only under at most *countable* unions and not necessarily under *uncountable* unions, we cannot conclude. But for a fixed $\omega \in \Omega$ such that $t \mapsto W_t(\omega)$ is continuous, if $W_s(\omega) > a$, there exists some $\tilde{s} \in \mathbb{Q} \cap [0,t]$ such that $W_{\tilde{s}}(\omega) > a$. Hence

$$\{W_s > a \text{ for some } s \in [0,t]\} = \{W_s > a \text{ for some } s \in \mathbb{Q} \cap [0,t]\} = \bigcup_{s \in \mathbb{Q} \cap [0,t]} \{W_s > a\}.$$

Note that $\{W_s > a\} \in \mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \in \mathbb{Q} \cap [0,t]$. Hence, since $\mathbb{Q} \cap [0,t]$ is now a countable set, we can conclude that $\bigcup_{s \in \mathbb{Q} \cap [0,t]} \{W_s > a\} \in \mathcal{F}_t$ and thus

$$\{\tau_a < t\} \in \mathcal{F}_t.$$

We can also express

$$\{\tau_a \leq t\} = \bigcap_{n=1}^{\infty} \{\tau_a < t + 1/n\} = \bigcap_{n=N}^{\infty} \{\tau_a < t + 1/n\} \quad \forall\, N \in \mathbb{N},$$

since $\{\tau_a < t + 1/(n+1)\} \subseteq \{\tau_a < t + 1/n\}$ for all $n \in \mathbb{N}$. Fix now an $N \in \mathbb{N}$ and observe that what we have shown before gives us that

$$\{\tau_a < t + 1/n\} \in \mathcal{F}_{t+1/n} \subseteq \mathcal{F}_{t+1/N}$$

for all $n \geq N$. Since σ-algebras are closed under countable intersections, we obtain that $\bigcap_{n=N}^{\infty} \{\tau_a < t+1/n\} \in \mathcal{F}_{t+1/N}$ and thus $\{\tau_a \leq t\} \in \mathcal{F}_{t+1/N}$ for all $N \in \mathbb{N}$. Since the filtration is right-continuous, we can conclude that

$$\{\tau_a \leq t\} \in \bigcap_{N \in \mathbb{N}} \mathcal{F}_{t+1/N} = \mathcal{F}_t.$$

For the second part, fix $a_1, a_2 \in \mathbb{R}$ such that $a_1 < a_2$. Note that for all $\omega \in \Omega$

$$\{t \geq 0 \mid W_t(\omega) > a_2\} \subseteq \{t \geq 0 \mid W_t(\omega) > a_1\}$$

and thus

$$\tau_{a_2} = \inf\{t \geq 0 \mid W_t > a_2\} \geq \inf\{t \geq 0 \mid W_t > a_1\} = \tau_{a_1} \quad P\text{-a.s.}$$

(b) By the global law of the iterated logarithm, we know that there exists an $\Omega_0 \subseteq \Omega$ such that $P[\Omega_0] = 1$ and

$$\limsup_{t \to \infty} \frac{W_t(\omega)}{\sqrt{2t \log(\log t)}} = 1 \quad \forall\, \omega \in \Omega_0. \tag{1}$$

Fix an $\omega \in \Omega_0$. Since $\lim_{t \to \infty} \sqrt{2t \log(\log t)} = \infty$, (1) implies that

$$\limsup_{t \to \infty} W_t(\omega) = \infty \quad \forall\, \omega \in \Omega_0.$$

Hence for all $x \in \mathbb{R}$ and $T > 0$, there exists a $t_x = t_x(\omega) > T$ such that $W_{t_x}(\omega) > x$. Choosing $x = a$, we see that $\tau_a(\omega) < t_x < \infty$. As a result $\tau_a(\omega) < \infty$ for all $\omega \in \Omega_0$ and thus

$$P[\tau_a < \infty] \geq P[\Omega_0] = 1.$$

(c) Fix an $a > 0$ and recall that $W_0 = 0 < a$ P-a.s. and thus, by continuity of Brownian motion, $\tau_a > 0$ P-a.s. By definition of τ_a, we have for all $\omega \in \Omega$ that

$$W_t(\omega) \leq a \quad \forall\, t < \tau_a(\omega).$$

Since $\tau_a > 0$ P-a.s., the set $[0, \tau_a)$ in nonempty P-a.s. and hence, again by continuity of Brownian motion,

$$W_{\tau_a(\omega)}(\omega) = \lim_{t \uparrow \tau_a(\omega)} W_t(\omega) \leq a,$$

for P-almost all $\omega \in \Omega$, i.e. $W_{\tau_a} \leq a$ P-a.s.

Fix now an $\omega \in \Omega_0$. By the definition of an infimum, there is a sequence

$$(t_n)_{n\in\mathbb{N}} \subseteq \{t \geq 0 \mid W_t(\omega) > a\}$$

such that $\lim_{n\to\infty} t_n = \tau_a(\omega)$. Observe that $W_{t_n}(\omega) > a$ for all $n \in \mathbb{N}$ and thus, by continuity,

$$W_{\tau_a(\omega)}(\omega) = \lim_{n\to\infty} W_{t_n}(\omega) \geq a.$$

Since the Brownian motion is continuous P-a.s., we obtain that $W_{\tau_a} \geq a$ P-a.s., and we can conclude that $W_{\tau_a} = a$ P-a.s.

We can now conclude that

$$E\left[W_{\tau_{a_2}} \,\middle|\, \mathcal{F}_{\tau_{a_1}}\right] = a_2 \neq a_1 = W_{\tau_{a_1}} \quad P\text{-a.s.}$$

We have thus proved that even for Brownian motion, a continuous $(P, \mathbb{F})$-martingale, there exist $\mathbb{F}$-stopping times σ, τ with $\sigma \leq \tau$ such that

$$E\left[W_\tau \,|\, \mathcal{F}_\sigma\right] \neq W_\sigma \quad P\text{-a.s.}$$

(d) As explained in (b), by the global law of the iterated logarithm there exists an $\Omega_0 \subseteq \Omega$ such that $P[\Omega_0] = 1$ and

$$\limsup_{t\to\infty} W_t(\omega) = \infty \quad \forall\, \omega \in \Omega_0.$$

Hence, for all $x \in \mathbb{R}$ and $T > 0$, there exists a $t_x = t_x(\omega) > T$ such that $W_{t_x}(\omega) > x$. Choosing $x = a$, we see that $\rho_a(\omega) \geq t_x > T$. As a result $\rho_a(\omega) = \infty$ for all $\omega \in \Omega_0$ and thus

$$P[\rho_a = \infty] \geq P[\Omega_0] = 1.$$

This directly implies that $P[\rho_a \leq t] = 0$ and thus, since $\mathbb{F}$ is P-complete, $\{\rho_a \leq t\} \in \mathcal{F}_0 \subseteq \mathcal{F}_t$ for all $t \geq 0$, proving that ρ_a is a stopping time.

Exercise 10.2 Let M be an RCLL local martingale null at 0 which satisfies $\sup_{0\leq s\leq T}|M_t| \in L^2$ for some $T \in \mathbb{R}$.

(a) Show that M is a square-integrable martingale on $[0, T]$.
Hint: Dominated convergence theorem.

(b) Let $[M]$ be the square bracket process of M. Show that

$$E\left[[M]_t - [M]_s \,|\, \mathcal{F}_s\right] = \mathrm{Var}[M_t - M_s \,|\, \mathcal{F}_s]$$

for all $0 \leq s \leq t \leq T$.

Solution 10.2

(a) Let us denote $M_T^* = \sup_{0\leq u\leq T}|M_u|$. The process M is adapted by definition since it is a local martingale. We also have that $M_s \in L^2$ for all $s \in [0, T]$ since we clearly have that $|M_s|^2 \leq |M_T^*|^2$ for all $s \in [0, T]$ and $M_T^* \in L^2$ by assumption, so M is square-integrable on $[0, T]$.

Now let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for M. We have for all $0 \leq s \leq t \leq T$ and all $n \in \mathbb{N}$ that

$$E\left[M_{\tau_n \wedge t} \,|\, \mathcal{F}_s\right] = M_{\tau_n \wedge s} \tag{2}$$

because M is a local martingale. Since $|M_{\tau_n \wedge t}|$ is bounded above for all $0 \le t \le T$ by an integrable random variable M_T^*, dominated convergence for conditional expectations gives us that

$$\lim_{n\to\infty} E\left[M_{\tau_n\wedge t} \,\middle|\, \mathcal{F}_s\right] = E\left[\lim_{n\to\infty} M_{\tau_n\wedge t} \,\middle|\, \mathcal{F}_s\right] = E\left[M_t \,\middle|\, \mathcal{F}_s\right]. \tag{3}$$

On the other hand, we have for the right-hand side of (2) that

$$\lim_{n\to\infty} M_{\tau_n\wedge s} = M_s,$$

which together with (3) gives us the martingale property for M on $[0,T]$ and concludes the proof.

(b) Since M is a square integrable martingale on $[0,T]$, the square bracket process $[M]$ is integrable and $M^2 - [M]$ is a martingale according to Theorem V.1.1 in the lecture notes. Therefore

$$\begin{aligned}
E\left[[M]_t - [M]_s \,\middle|\, \mathcal{F}_s\right] &= E\left[M_t^2 - M_s^2 \,\middle|\, \mathcal{F}_s\right] \\
&= E\left[M_t^2 - 2M_tM_s + M_s^2 + 2M_tM_s - 2M_s^2 \,\middle|\, \mathcal{F}_s\right] \\
&= E\left[(M_t - M_s)^2 \,\middle|\, \mathcal{F}_s\right] + 2M_sE\left[M_t - M_s \,\middle|\, \mathcal{F}_s\right] \\
&= E\left[(M_t - M_s)^2 \,\middle|\, \mathcal{F}_s\right] = E\left[(M_t - E[M_t \,|\, \mathcal{F}_s])^2 \,\middle|\, \mathcal{F}_s\right] \\
&= E\left[M_t - M_s + M_s - E[M_t \,|\, \mathcal{F}_s])^2 \,\middle|\, \mathcal{F}_s\right] \\
&= E\left[\Big((M_t - M_s) - E[M_t - M_s \,|\, \mathcal{F}_s]\Big)^2 \,\middle|\, \mathcal{F}_s\right] \\
&= \mathrm{Var}[M_t - M_s \,|\, \mathcal{F}_s]
\end{aligned}$$

for all $0 \le s \le t \le T$.

Exercise 10.3 On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, consider an adapted stochastic process $X = (X_t)_{t\ge 0}$ null at 0. Assume that it is integrable and has independent stationary increments, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same distribution as X_{t-s} for all $t > s$. (In particular, this is satisfied for any *Lévy process* $L = (L_t)_{t\ge 0}$ with $E[|L_1|] < \infty$).

(a) What conditions must $\big(E[X_t]\big)_{t\ge 0}$ satisfy in order to make X a $(P,\mathbb{F})$-supermartingale, a $(P,\mathbb{F})$-submartingale, or a $(P,\mathbb{F})$-martingale?

(b) Assume from now on that X is a square-integrable $(P,\mathbb{F})$-martingale. Prove that we have for all $t, s > 0$ that

$$E\left[X_t^2\right] + E\left[X_s^2\right] = E\left[X_{t+s}^2\right]$$

and deduce that $\big(E\left[X_t^2\right]\big)_{t\ge 0}$ is an increasing process.

(c) Use (b) to prove that $E\left[X_t^2\right] = tE\left[X_1^2\right]$ for all $t \ge 0$.
Hint: Prove the result first for $t = 1/n$ for all $n \in \mathbb{N}$. Deduce that it holds true for all $t \in \mathbb{Q}_+$ and use monotonicity to conclude.

(d) Prove that $\langle X \rangle_t = tE\left[X_1^2\right]$, for all $t \ge 0$.
Hint: Use your result from (c).

Solution 10.3

(a) Adaptedness and integrability are already given by assumption. We can then use that X has independent stationary increments to compute

$$E[X_t \,|\, \mathcal{F}_s] = E[X_t - X_s \,|\, \mathcal{F}_s] + X_s = E[X_t - X_s] + X_s = E[X_{t-s}] + X_s,$$

for all $t > s$. As a result X is a $(P,\mathbb{F})$-supermartingale if and only if $E[X_t] \le 0$ for all $t \ge 0$, a $(P,\mathbb{F})$-submartingale if and only if $E[X_t] \ge 0$ for all $t \ge 0$, and a $(P,\mathbb{F})$-martingale if and only if $E[X_t] = 0$ for all $t \ge 0$.

(b) By the martingale property of X and the stationarity of the increments, we can directly compute

$$\begin{aligned} E\left[X_{t+s}^2\right] - E\left[X_t^2\right] = E\left[X_{t+s}^2 - X_t^2\right] &= E\left[(X_{t+s} - X_t)^2\right] + 2E\left[X_t\big(X_{t+s} - X_t\big)\right] \\ &= E\left[(X_{t+s} - X_t)^2\right] + 2E\left[X_t E\left[X_{t+s} - X_t \,\middle|\, \mathcal{F}_t\right]\right] = E\left[X_s^2\right], \end{aligned}$$

for all $t, s > 0$. As a result, for $t > s$ we have that

$$E\left[X_t^2\right] - E\left[X_s^2\right] = E\left[X_{t-s}^2\right] \geq 0,$$

proving that the process $\left(E\left[X_t^2\right]\right)_{t\geq 0}$ is increasing.

(c) Let first $t = 1/n$ for some $n \in \mathbb{N}$. We want to show that $nE\left[X_{1/n}^2\right] = E\left[X_1^2\right]$. We compute

$$nE\left[X_{1/n}^2\right] = \sum_{k=1}^{n} E\left[X_{1/n}^2\right] = \sum_{k=1}^{n} \left(E\left[X_{k/n}^2\right] - E\left[X_{(k-1)/n}^2\right]\right) = E\left[X_1^2\right] - E\left[X_0^2\right] = E\left[X_1^2\right],$$

where in the second equality we have used our result from (b). If we now consider an arbitrary number $\ell/n \in \mathbb{Q}_+$, we can use the same technique as in the above to compute

$$\ell E\left[X_{1/n}^2\right] = \sum_{k=1}^{\ell} E\left[X_{1/n}^2\right] = E\left[X_{\ell/n}^2\right].$$

Since we have that $E\left[X_{1/n}^2\right] = \frac{1}{n}E\left[X_1^2\right]$, we can conclude that $E\left[X_{\ell/n}^2\right] = \frac{\ell}{n}E\left[X_1^2\right]$. We have thus proved that $E\left[X_t^2\right] = tE\left[X_1^2\right]$ for all $t \in \mathbb{Q}_+$, but since $E\left[X_t^2\right]$ is increasing, we can compute

$$tE\left[X_1^2\right] = \sup_{s\in\mathbb{Q}_+, s<t} sE\left[X_1^2\right] = \sup_{s\in\mathbb{Q}_+, s<t} E\left[X_s^2\right] \leq E\left[X_t^2\right] \leq \inf_{s\in\mathbb{Q}_+, s>t} E\left[X_s^2\right] = tE\left[X_1^2\right],$$

and thus conclude that $E\left[X_t^2\right] = tE\left[X_1^2\right]$.

(d) We first have to show that $\left(tE\left[X_1^2\right]\right)_{t\geq 0}$ is an increasing, predictable process null at 0. Since $E\left[X_1^2\right] \geq 0$, the process is clearly increasing. Moreover, since it is deterministic, it is also predictable and it is clearly null at 0. It only remains to show that the process $\left(X_t^2 - tE\left[X_1^2\right]\right)_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale. Since the increments are independent and stationary, for all $t > s$ we can compute

$$\begin{aligned} E\left[X_t^2 - X_s^2 \,\middle|\, \mathcal{F}_s\right] &= E\left[(X_t - X_s)^2 \,\middle|\, \mathcal{F}_s\right] + 2X_s E\left[X_t - X_s \,\middle|\, \mathcal{F}_s\right] \\ &= E\left[(X_t - X_s)^2\right] = E\left[X_{t-s}^2\right] = (t-s)E\left[X_1^2\right] \\ &= tE\left[X_1^2\right] - sE\left[X_1^2\right], \end{aligned}$$

where the fourth equality uses our result from (c). Rearranging the above, we obtain that

$$E\left[X_t^2 - tE\left[X_1^2\right] - \left(X_s^2 - sE\left[X_1^2\right]\right) \,\middle|\, \mathcal{F}_s\right] = 0,$$

which is the martingale property for $\left(X_t^2 - tE\left[X_1^2\right]\right)_{t\geq 0}$.