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Mathematical Foundations for Finance

Exercise sheet 11

Exercise 11.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. Assume that $\mathcal{F}_0$ is P-trivial and consider a Brownian motion W on this space.

(a) Prove that any continuous, adapted process H is predictable and locally bounded.
Hint 1: Recall that a process X is locally bounded if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to infinity such that each X^{τ_n} is uniformly bounded P-a.s.

(b) Prove that any predictable, locally bounded process H is an element of $L^2_{\text{loc}}(W)$.
Hint: We saw that $L^2_{loc}(M)$ can be characterized in a nice way when M is a continuous local martingale null at 0.

(c) Deduce that for any function $f : \mathbb{R} \to \mathbb{R}$ in C^1, the stochastic integral $\int_0^\cdot f'(W_s)dW_s$ is a continuous local martingale.

(d) Conclude using Itô's formula that $f(W)$ for a given $f \in C^2$ is a continuous local martingale if and only if $\int_0^\cdot f''(W_s)ds = 0$.
Hint 1: If M and N are local $(P, \mathbb{F})$-martingales, then $M + N$ is a local $(P, \mathbb{F})$-martingale.
Hint 2: For every continuous local martingale M null at 0 and with finite variation, we have that $M = 0$ P-a.s.

Solution 11.1

(a) Recall that a process H is predictable if it is $\mathcal{P}$-measurable when viewed as a mapping $H : \overline{\Omega} \to \mathbb{R}$, for $\overline{\Omega} := \Omega \times (0, \infty)$ and $\mathcal{P}$ being the σ-field on $\overline{\Omega}$ generated by all left-continuous adapted processes. Since H is adapted and continuous (therefore also left-continuous), it is obviously predictable.

Define now $(\tau_n)_{n\in\mathbb{N}}$ by

$$\tau_n := \inf\{t \geq 0 \,|\, |H_t| > n\}$$

for all $n \in \mathbb{N}$. Observe that τ_n is a stopping time for all $n \in \mathbb{N}$, by the continuity of H and the right-continuity of the filtration (see also Exercise 10.1 (a) where we have done this for BM, and realize that the proof uses only the continuity of BM). The sequence $(\tau_n)_{n\in\mathbb{N}}$ is then clearly increasing P-a.s.

Fix now an $\omega \in \Omega$ such that the map $t \mapsto H_t(\omega)$ is continuous. Since continuous functions are bounded on compact intervals, we have that for all $T \geq 0$, there exists an $N := N(\omega, T) \in \mathbb{N}$ such that $|H_t(\omega)| < N$ for all $t \in [0, T]$, and thus $\tau_n(\omega) \geq T$ for all $n \geq N$. As a result $\lim_{n\to\infty} \tau_n(\omega) = \infty$ and hence $\lim_{n\to\infty} \tau_n = \infty$ P-a.s. We can thus conclude that $(\tau_n)_{n\in\mathbb{N}}$ defines a localizing sequence.

Finally, by the definition of τ_n, we have that for all $\omega \in \Omega$,

$$|H_t(\omega)| \leq n \qquad \forall\ t < \tau_n(\omega).$$

There are now two possible cases. Either $\tau_n(\omega) = 0$ and hence $|H_{\tau_n(\omega)}(\omega)| = |H_0(\omega)|$, or $\tau_n(\omega) > 0$ in which case $[0, \tau_n(\omega)) \neq \emptyset$ and by continuity of H we can compute

$$|H_{\tau_n(\omega)}(\omega)| = \lim_{\substack{t\to\tau_n(\omega) \\ t<\tau_n(\omega)}} |H_t(\omega)| \leq n$$

for P-a.a. $\omega \in \Omega$. Since $\mathcal{F}_0$ is P-trivial, $H_0 = h_0 \in \mathbb{R}$ P-a.s. and we can conclude that $|H_t(\omega)| \leq n \vee |h_0|$ for all $t \leq \tau_n(\omega)$ and P-a.a. $\omega \in \Omega$ and thus

$$|H_t^{\tau_n}| \leq n \vee |h_0| \quad P\text{-a.s. for all } t \geq 0.$$

(b) Since W is a *continuous* (local) martingale, $H \in L^2_{\text{loc}}(W)$ if and only if it is predictable and

$$\int_0^t H_s^2 ds = \int_0^t H_s^2 d[W]_s < \infty \qquad P\text{-a.s.}$$

for each $t \geq 0$ (see top of the page 87 in the lecture notes). The first property is true by assumption. For the second one, let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times increasing P-a.s. to infinity such that H^{τ_n} is uniformly bounded P-a.s. (i.e. $|H_t^{\tau_n}| \leq c_n$ for some $c_n \geq 0$ for all $t \geq 0$). Let Ω_0 be the set of all $\omega \in \Omega$ such that $\lim_{n\to\infty} \tau_n(\omega) = \infty$ and $|H_t^{\tau_n}(\omega)| \leq c_n$ for all $t \geq 0$ and $n \in \mathbb{N}$. Since countable intersections of sets of probability one are of probability 1, $P[\Omega_0] = 1$. Fix then an $\omega \in \Omega_0$ and a $t > 0$. Observe that since $\lim_{n\to\infty} \tau_n(\omega) = \infty$, there exists an $N := N(\omega, t) \in \mathbb{N}$ such that $\tau_N(\omega) > t$. As a result

$$\int_0^t H_s^2(\omega) ds = \int_0^t \left(H_s^{\tau_N}(\omega)\right)^2 ds \leq \int_0^t c_N^2 ds = c_N^2 t < \infty$$

and hence $\int_0^t H_s^2(\omega) ds < \infty$ for all $\omega \in \Omega_0$ and $t > 0$.

(c) First note that $\left(f'(W_t)\right)_{t\geq 0}$ is adapted and continuous because f' is continuous and W is adapted and continuous. By (a), $f'(W)$ is then predictable and locally bounded. Since $f'(W) \in L^2_{\text{loc}}(W)$ by (b) and since W is a (local) martingale null at 0, the stochastic integral $\int_0^\cdot f'(W_s) dW_s$ is well defined by and is indeed a continuous local martingale null at 0 (see the bottom of page 86 in the lecture notes).

(d) Since $f \in C^2$, we can compute by Itô's lemma that

$$f(W_t) = f(0) + \int_0^t f'(W_s) dW_s + \frac{1}{2} \int_0^t f''(W_s) ds. \tag{1}$$

Since $\int_0^\cdot f'(W_s) dW_s$ is a continuous local martingale by (c), it is clear that the same holds true for $f(W)$ if $\int_0^t f''(W_s) ds = 0$ P-a.s. for all $t \geq 0$.

Assume now for the converse that $f(W)$ is a continuous local martingale. We obtain by rearranging (1) that

$$\int_0^t f''(W_s) ds = 2f(W_t) - 2f(0) - 2\int_0^t f'(W_s) dW_s \qquad P\text{-a.s. for all } t \geq 0.$$

Since the right-hand side is a sum of two local $(P, \mathbb{F})$-martingales and it is obviously null at 0, we can conclude by Hint 1 that $\int_0^\cdot f''(W_s) ds$ is a local martingale null at 0.

However, $\int_0^\cdot f''(W_s(\omega)) d\langle W\rangle_s(\omega) = \int_0^\cdot f''(W_s(\omega)) ds$ is defined pathwise for P-a.a. $\omega \in \Omega$ as a Lebesgue-Stieltjes integral. Since $f''(W_s(\omega))$ is continuous (as a function of s) for P-a.a. $\omega \in \Omega$, it is also P-a.s. bounded on any compact interval $[0, t]$ and therefore integrable on any such interval. But this means that the paths

$$g_\omega(t) = \int_0^t f''(W_s(\omega)) ds$$

are absolutely continuous for P-a.a. $\omega \in \Omega$ and hence of finite variation for P-a.a. $\omega \in \Omega$. We can therefore conclude by Hint 2 that $\int_0^\cdot f''(W_s) ds$ is identically equal to 0 P-a.s.

Alternatively, we could compute for a any partition Π of $[0,\infty)$

$$\begin{aligned}\sum_{t_i\in\Pi}|g_\omega(t_i\wedge t)-g_\omega(t_{i-1}\wedge t)| &= \sum_{t_i\in\Pi}\left|\int_{t_{i-1}\wedge t}^{t_i\wedge t} f''(W_s(\omega))ds\right| \le \sum_{t_i\in\Pi}\int_{t_{i-1}\wedge t}^{t_i\wedge t}|f''(W_s(\omega))|\,ds\\ &= \int_0^{t_N\wedge t}|f''(W_s(\omega))|\,ds \le \int_0^t |f''(W_s(\omega))|\,ds,\end{aligned}$$

where the last expression is finite for P-a.a. $\omega\in\Omega$ since for P-a.a. $\omega\in\Omega$ it is an integral of a continuous function on a compact interval. Since the above holds for any partition Π of $[0,\infty)$ we must also have that

$$\sup_\Pi \sum_{t_i\in\Pi}|g_\omega(t_i\wedge t)-g_\omega(t_{i-1}\wedge t)| \le \int_0^t |f''(W_s(\omega))|\,ds < \infty,$$

and hence we can conclude using Hint 2 as before.

Exercise 11.2 Let $W=(W_t)_{t\ge0}$ be a Brownian motion with respect to a probability measure P and a filtration $\mathbb{F}=(\mathcal{F}_t)_{t\ge0}$. Using Itô's formula, decide for each of the following processes whether they are local $(P,\mathbb{F})$-martingales or not. Which of them are even $(P,\mathbb{F})$-martingales?

(a) $X_t^{(1)} := \exp\left(\frac12\alpha^2 t\right)\cos\left(\alpha(W_t-\beta)\right)$, $t\ge0$, where $\alpha,\beta\in\mathbb{R}$.
Hint: For the martingale property of $X^{(1)}$, look first at $[0,T]$ for some $T>0$.

(b) $X_t^{(2)} := \sin W_t - \cos W_t$, $t\ge0$.

(c) $X_t^{(3)} := W_t^p - ptW_t$, $t\ge0$, for $p\in\mathbb{N}$ with $p\ge2$.
Hint: For any $T>0$, $\sup_{0\le t\le T} W_t$ has the same distribution as $|W_T|$ and so has $-\inf_{0\le s\le T} W_s$.

Solution 11.2 First, we notice that $X^{(1)}, X^{(2)}, X^{(3)}$ are all of the form $X_t = f(W_t,t)$ with $f:\mathbb{R}^2\to\mathbb{R}$ in C^2. By Itô's formula, it then follows that

$$X_t = X_0 + \int_0^t \frac{\partial f}{\partial w}(W_s,s)dW_s + \int_0^t\left(\frac{\partial f}{\partial t}(W_s,s)+\frac12\frac{\partial^2 f}{(\partial w)^2}(W_s,s)\right)ds.$$

Since W is a continuous $(P,\mathbb{F})$-martingale and since the integrand $\left(\frac{\partial f}{\partial w}(W_t,t)\right)_{t\ge0}$ is continuous and adapted, and therefore also predictable, the integrand is also an element of $L^2_{\text{loc}}(W)$ and we have that

$$\int_0^\cdot \frac{\partial f}{\partial w}(W_s,s)dW_s$$

is a local $(P,\mathbb{F})$-martingale. Moreover, analogously to Exercise 11.1 (d), the process X is a (continuous) local $(P,\mathbb{F})$-martingale if and only if

$$\int_0^t\left(\frac{\partial f}{\partial t}(W_s,s)+\frac12\frac{\partial^2 f}{(\partial w)^2}(W_s,s)\right)ds = 0,\qquad P\text{-a.s. for all } t\ge0. \tag{2}$$

(a) We have $X_t^{(1)} = f^{(1)}(W_t,t)$, where $f^{(1)}(w,t)=\exp\left(\frac12\alpha^2t\right)\cos\left(\alpha(w-\beta)\right)$. A direct computation shows that

$$\frac{\partial f^{(1)}}{\partial t}+\frac12\frac{\partial^2 f^{(1)}}{(\partial w)^2} = 0\quad\text{in } \mathbb{R}\times(0,\infty).$$

Hence, by (2), $X^{(1)}$ is a local $(P,\mathbb{F})$-martingale. But for any $T\ge0$, the process $X^{(1)}$ is bounded on $[0,T]$, i.e.

$$\left|X_t^{(1)}\right| \le \exp\left(\frac12\alpha^2T\right)\quad\text{for all } t\in[0,T], \tag{3}$$

which means, in particular, that $X^{(1)}$ is integrable on $[0,T]$. Now let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for $X^{(1)}$. By (3), dominated convergence theorem gives us that

$$E\left[X_t^{(1)}\,\middle|\,\mathcal{F}_s\right] = E\left[\lim_{n\to\infty} X_{t\wedge\tau_n}^{(1)}\,\middle|\,\mathcal{F}_s\right] = \lim_{n\to\infty} E\left[X_{t\wedge\tau_n}^{(1)}\,\middle|\,\mathcal{F}_s\right] = \lim_{n\to\infty} X_{s\wedge\tau_n}^{(1)} = X_s^{(1)},$$

which is the martingale property of $X^{(1)}$ on $[0,T]$ and concludes showing that $X^{(1)}$ is a true $(P,\mathbb{F})$-martingale on $[0,T]$. However, the above holds true for any $T>0$ and since we can write $[0,\infty)=\bigcup_{T>0}[0,T]$, $X^{(1)}$ is in fact a true $(P,\mathbb{F})$-martingale on $[0,\infty)$.

(b) We have $X_t^{(2)} = f^{(2)}(W_t,t)$, where $f^{(2)}(w,t)=\sin(w)-\cos(w)$. A direct computation shows that

$$\frac{\partial f^{(2)}}{\partial t}(w,t) + \frac{1}{2}\frac{\partial^2 f^{(2)}}{(\partial w)^2}(w,t) = \frac{1}{2}(\cos w - \sin w). \tag{4}$$

Observe that for $w\in[-\pi/8,\pi/8]$ we have that the expression in (4) is strictly positive. So, $X^{(2)}$ is not a local $(P,\mathbb{F})$-martingale and consequently not a $(P,\mathbb{F})$-martingale.

(c) We have $X_t^{(3)} = f^{(3)}(W_t,t)$, where $f^{(3)}(w,t)=w^p-ptw$. Moreover

$$\frac{\partial f^{(3)}}{\partial t}(w,t) + \frac{1}{2}\frac{\partial^2 f^{(3)}}{(\partial w)^2}(w,t) = -pw + \frac{1}{2}p(p-1)w^{p-2}, \quad (w,t)\in\mathbb{R}\times(0,\infty).$$

The latter term is identically equal to 0 if and only if $p=3$. Hence, by (2), the process $X^{(3)}$ is a local $(P,\mathbb{F})$-martingale if and only if $p=3$.

In order to show that $X^{(3)}$ is indeed a true $(P,\mathbb{F})$-martingale, we will use the result from Exercise 10.2 (a), i.e. that if $M=(M_t)_{t\geq 0}$ is an RCLL local $(P,\mathbb{F})$-martingale null at 0 with $\sup_{0\leq t\leq T}|M_t|\in L^2(P)$, then M is a true $(P,\mathbb{F})$-martingale on $[0,T]$.

Since $\sup_{0\leq t\leq T} W_t \geq 0$, we can write for all $T>0$ that

$$\sup_{0\leq t\leq T}|W_t| \leq \sup_{0\leq t\leq T} W_t + \left|\inf_{0\leq t\leq T} W_t\right| = \sup_{0\leq t\leq T} W_t + \left|-\inf_{0\leq t\leq T} W_t\right|.$$

But since we have by the hint that $\sup_{0\leq t\leq T} W_t \overset{(d)}{=} |W_T|$ and $-\inf_{0\leq s\leq T} W_t \overset{(d)}{=} |W_T|$, we have, in particular, that $\sup_{0\leq t\leq T} W_t$ and $|-\inf_{0\leq t\leq T} W_t|$ both belong to $L^q(P)$ for any $q\in(0,\infty)$. We thus also have that $\sup_{0\leq t\leq T}|W_t|\in L^q(P)$ for any $q\in(0,\infty)$.

Moreover, by the above result, we have that

$$\sup_{0\leq t\leq T}\left|X_t^{(3)}\right| \leq \left(\sup_{0\leq t\leq T}|W_t|\right)^3 + 3T\sup_{0\leq t\leq T}|W_t| \in L^2(P)$$

for all $T>0$, and hence on $[0,\infty)$, and we are done.

Exercise 11.3 Let $W=(W_t)_{t\geq 0}$ be a Brownian motion with respect to some probability measure P and a filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$. Use Itô's formula to write the following processes as stochastic integrals.

(a) $X_t^{(1)} = W_t^2$.

(b) $X_t^{(2)} = t^2 W_t^3$.

(c) $X_t^{(3)} = \exp(mt+\sigma W_t)$.

(d) $X_t^{(4)} = \cos(t+W_t)$.

(e) $X_t^{(5)} = \log\left(2 + \cos\left(W_t - t\right)\right)$.

(f) Let X and Y be two continuous real-valued $(P, \mathbb{F})$-semimartingales. Define the process $Z = XY$. Apply Itô's formula to Z and write it as a sum of stochastic integrals.

Solution 11.3 First we note that the value at time t of the first five processes can be written as the value of some C^2-functions at the point (t, W_t). The process $((t, W_t))_{t\geq 0}$ is a continuous semimartingale, so we can apply Itô's formula.

(a) $X_t^{(1)} = W_t^2$. We have $X_t^{(1)} = f^{(1)}(W_t)$ for $f^{(1)} : x \mapsto x^2$. A quick computation gives

$$f_x^{(1)}(x) = 2x, \qquad f_{xx}^{(1)}(x) = 2,$$

and by Itô's formula, we obtain

$$\begin{aligned} X_t^{(1)} &= X_0^{(1)} + \int_0^t f_x^{(1)}(W_s) dW_s + \frac{1}{2}\int_0^t f_{xx}^{(1)}(W_s) ds \\ &= 2\int_0^t W_s dW_s + \int_0^t ds \\ &= 2\int_0^t W_s dW_s + t. \end{aligned}$$

Note that this also gives that

$$\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t$$

as shown in a different way in the example on the page 89 in the lecture notes.

(b) We have $X_t^{(2)} = f^{(2)}(t, W_t)$ for $f^{(2)} : (t, x) \mapsto t^2 x^3$. Partial differentiation gives

$$f_t^{(2)}(t,x) = 2tx^3, \qquad f_{t,x}^{(2)}(x) = 3t^2x^2, \qquad f_{xx}^{(2)}(t,x) = 6t^2x.$$

By Itô's formula, we get that

$$\begin{aligned} X_t^{(2)} &= X_0^{(2)} + \int_0^t f_t^{(2)}(s, W_s) ds + \int_0^t f_x^{(2)}(s, W_s) dW_s + \frac{1}{2}\int_0^t f_{xx}^{(2)}(s, W_s) ds \\ &= \int_0^t \left(2sW_s^3 + 3s^2 W_s\right) ds + 3\int_0^t s^2 W_s^2 dW_s. \end{aligned}$$

(c) We can write $X_t^{(3)} = f^{(3)}(t, W_t)$ for $f^{(3)} : (t, x) \mapsto \exp\left(mt + \sigma W_t\right)$. We compute

$$f_t^{(3)}(t,x) = m\exp\left(mt + \sigma x\right), \;\; f_x^{(3)}(t,x) = \sigma\exp\left(mt + \sigma x\right), \;\; f_{xx}^{(3)}(t,x) = \sigma^2\exp\left(mt + \sigma x\right).$$

As before, Itô's formula yields

$$\begin{aligned} X_t^{(3)} &= X_0^{(3)} + \int_0^t f_t^{(3)}(s, W_s) ds + \int_0^t f_x^{(3)}(s, W_s) dW_s + \frac{1}{2}\int_0^t f_{xx}^{(3)}(s, W_s) ds \\ &= 1 + \int_0^t \left(m + \frac{1}{2}\sigma^2\right)\exp\left(ms + \sigma W_s\right) ds + \int_0^t \sigma\exp\left(ms + \sigma W_s\right) dW_s. \end{aligned}$$

We can also rewrite the last equality as

$$X_t^{(3)} - X_0^{(3)} = \int_0^t \left(m + \frac{1}{2}\sigma^2\right) X_s^{(3)} ds + \int_0^t \sigma X_s^{(3)} dW_s.$$

(d) Define $f^{(4)} : (t,x) \mapsto \cos(t+x)$; then $X_t^{(4)} = f^{(4)}(t, W_t)$. We need to compute the three partial derivatives

$$f_t^{(4)}(t,x) = -\sin(t+x)\,, \qquad f_x^{(4)}(t,x) = -\sin(t+x)\,, \qquad f_{xx}^{(4)}(t,x) = -\cos(t+x)\,.$$

Itô's formula yields

$$\begin{aligned} X_t^{(4)} &= X_0^{(4)} + \int_0^t f_t^{(4)}(s,W_s)ds + \int_0^t f_x^{(4)}(s,W_s)dW_s + \frac{1}{2}\int_0^t f_{xx}^{(4)}(s,W_s)ds \\ &= 1 - \int_0^t \left(\sin(s+W_s) + \frac{1}{2}\cos(s+W_s)\right)ds - \int_0^t \sin(s+W_s)\,dW_s. \end{aligned}$$

(e) Define $f^{(5)} : (t,x) \mapsto \log(2+\cos(x-t))$; then $X_t^{(5)} = f^{(5)}(t, W_t)$. We compute

$$\begin{aligned} f_t^{(5)}(t,x) &= \frac{\sin(x-t)}{2+\cos(x-t)}, \qquad f_x^{(5)}(t,x) = -\frac{\sin(x-t)}{2+\cos(x-t)}, \\ f_{xx}^{(5)}(t,x) &= -\frac{\cos(x-t)(2+\cos(x-t)) + \sin(x-t)^2}{(2+\cos(x-t))^2} \\ &= -\frac{1+2\cos(x-t)}{(2+\cos(x-t))^2}. \end{aligned}$$

By Itô's formula, we can write

$$\begin{aligned} X_t^{(5)} &= X_0^{(5)} + \int_0^t f_t^{(5)}(s,W_s)ds + \int_0^t f_x^{(5)}(s,W_s)dW_s + \frac{1}{2}\int_0^t f_{xx}^{(5)}(s,W_s)ds \\ &= \log(3) + \int_0^t \left(\frac{\sin(W_s-s)}{2+\cos(W_s-s)} - \frac{1+2\cos(W_s-s)}{2(2+\cos(W_s-s))^2}\right)ds - \int_0^t \frac{\sin(W_s-s)}{2+\cos(W_s-s)}dW_s. \end{aligned}$$

(f) We can write $Z_t = g(X_t, Y_t)$ for $g : (x,y) \mapsto xy$, which is twice continuously differentiable in every variable. Applying Itô's formula to g, we get that

$$Z_t - Z_0 = \int_0^t Y_s dX_s + \int_0^t X_s dY_s + \int_0^t d[X,Y]_s\,,$$

since X and Y are continuous. This formula is sometimes referred to as the *product formula* or the *integration by parts formula*.