

Mathematical Foundations for Finance

Exercise sheet 12

Please hand in your solutions until Tuesday, 11/12/2018, 18:00 into your assistant's box next to HG G 53.2.

Exercise 12.1 In this exercise, we show various results that are frequently used in stochastic analysis. Some of them were given as hints in the previous exercises.

- (a) Let X be an RCLL \mathbb{F} -adapted stochastic process and τ an \mathbb{F} -stopping time. Show that if X^τ is an \mathbb{F} -martingale, then so is X^σ for any \mathbb{F} -stopping time σ with $\sigma \leq \tau$ P -a.s.
Hint: You can use the result that a stopped RCLL martingale is again an RCLL martingale. This is similar to the result you have proved in Exercise 3.1 (c).
- (b) Let M and N be two RCLL local \mathbb{F} -martingales. Show that the linear combination $\alpha M + \beta N$ for any $\alpha, \beta \in \mathbb{R}$ is an RCLL local \mathbb{F} -martingale as well.
Hint: Make use of the result in (a).
- (c) We say that two Brownian motions W^1 and W^2 on the same probability space (Ω, \mathcal{F}, P) endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ are *correlated with correlation* $\rho \in [-1, 1]$ if for $s \leq t$, the increments $W_t^1 - W_s^1$ and $W_t^2 - W_s^2$ are independent of \mathcal{F}_s and jointly normally distributed with $\mathcal{N}(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Sigma = \begin{pmatrix} t-s & \rho(t-s) \\ \rho(t-s) & t-s \end{pmatrix}.$$

Show that $[W^1, W^2]_t = \rho t$ P -a.s.

Hint: Define $B^\lambda = \lambda(W^1 + W^2)$ with $\lambda \in \mathbb{R}$. Find λ such that B^λ becomes a (P, \mathbb{F}) -Brownian motion. Then compute $[B^\lambda]$ in terms of W^1 and W^2 , using the properties of $[\cdot, \cdot]$.

Exercise 12.2 Let $X = (X_t)_{t \geq 0}$ be a continuous semimartingale null at 0. We define the process

$$L := \mathcal{E}(X) := e^{X - \frac{1}{2}[X]}.$$

- (a) Show via Itô's formula that

$$L_t = 1 + \int_0^t L_s dX_s, \quad \forall t \geq 0. \quad (1)$$

Conclude that L is a continuous local martingale if and only if X is a continuous local martingale.

- (b) Show that $L = \mathcal{E}(X)$ is the only solution to (1) for a given X .

Hint: Let L' be another solution of (1). Compute $\frac{L'}{L}$ using Itô's formula.

- (c) Let $Y = (Y_t)_{t \geq 0}$ be another continuous semimartingale null at 0. Show *Yor's formula*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y]).$$

Exercise 12.3 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ be a filtered probability space and consider two *independent* Brownian motions $W^1 = (W_t^1)_{t \in [0, T]}$ and $W^2 = (W_t^2)_{t \in [0, T]}$. Let $\tilde{S}^1 = (\tilde{S}_t^1)_{t \in [0, T]}$ and $\tilde{S}^2 = (\tilde{S}_t^2)_{t \in [0, T]}$ be two *undiscounted* stock price processes with the dynamics

$$\begin{aligned} d\tilde{S}_t^1 &= \tilde{S}_t^1 (\mu_1 dt + \sigma_1 dB_t^1), & \tilde{S}_0^1 &> 0, \\ d\tilde{S}_t^2 &= \tilde{S}_t^2 (\mu_2 dt + \sigma_2 dB_t^2), & \tilde{S}_0^2 &> 0, \end{aligned}$$

where $B^1 = W^1$, $B^2 = \alpha W^1 + \sqrt{1 - \alpha^2} W^2$, for some $\alpha \in [0, 1)$, $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$.

- (a) Find the SDEs satisfied by $X^1 := \frac{\tilde{S}^2}{\tilde{S}^1}$ and $X^2 := \frac{\tilde{S}^1}{\tilde{S}^2}$.

Remark: Since \tilde{S}^1 and \tilde{S}^2 have continuous trajectories and satisfy $\tilde{S}_t^1, \tilde{S}_t^2 > 0$ for all $t \in [0, T]$ P -a.s., we can choose each of them as *numéraire*.

- (b) For $\beta_1, \beta_2 \in \mathbb{R}$, define the continuous local (P, \mathbb{F}) -martingale $L^{(\beta_1, \beta_2)} := \beta_1 W^1 + \beta_2 W^2$. Show that for all $\beta_1, \beta_2 \in \mathbb{R}$, the stochastic exponential $Z^{(\beta_1, \beta_2)} := \mathcal{E}(L^{(\beta_1, \beta_2)})$ is a true (P, \mathbb{F}) -martingale on $[0, T]$.

- (c) For $\beta_1, \beta_2 \in \mathbb{R}$, define by $dQ^{(\beta_1, \beta_2)} = Z_T^{(\beta_1, \beta_2)} dP$ a probability measure $Q^{(\beta_1, \beta_2)}$ which is equivalent to P on \mathcal{F}_T . Fix $\beta_1, \beta_2 \in \mathbb{R}$. Using Girsanov's theorem, show that the two processes $\tilde{W}_t^1 := W_t^1 - \beta_1 t$ and $\tilde{W}_t^2 := W_t^2 - \beta_2 t$, $t \in [0, T]$, are local $(Q^{(\beta_1, \beta_2)}, \mathbb{F})$ -martingales. Conclude that

$$\tilde{B}^1 := \tilde{W}^1 \quad \text{and} \quad \tilde{B}_t^2 := B_t^2 - (\alpha\beta_1 + \sqrt{1 - \alpha^2}\beta_2)t, \quad t \in [0, T],$$

are local $(Q^{(\beta_1, \beta_2)}, \mathbb{F})$ -martingales as well.

Remark: One can show that \tilde{W}^1 and \tilde{W}^2 are *independent* Brownian motions under $Q^{(\beta_1, \beta_2)}$ and correspondingly that \tilde{B}^1 and \tilde{B}^2 are *correlated* Brownian motions under $Q^{(\beta_1, \beta_2)}$.

- (d) What conditions on $\beta_1, \beta_2 \in \mathbb{R}$ make the processes X^1 and X^2 $(Q^{(\beta_1, \beta_2)}, \mathbb{F})$ -martingales?