ETH zürich

Mathematical Foundations for Finance

Exercise 12

Martin Stefanik
ETH Zurich

Itô Representation Theorem

Let $\mathbb{F}^W = (\mathcal{F}_t^W)_{0\leq t\leq\infty}$ denote the filtration generated by a Brownian motion W augmented by P-nullsets in $\mathcal{F}_\infty^0 = \sigma(W_s, s \geq 0)$.

Theorem 1 (Itô representation theorem)

Suppose that $W = (W_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^m$, $m \in \mathbb{N}$. Then every random variable $H \in L^1(\mathcal{F}_\infty^W, P)$ has a unique *representation as*

$$H = E[H] + \int_0^\infty \psi_s dW_s \quad P\text{-a.s.}$$

for an $\mathbb{R}^m$-valued integrand $\psi \in L^2_{loc}(W)$ with the additional property that $\int \psi dW$ is a $(P, \mathbb{F}^W)$-martingale on $[0, \infty]$.

- We can also make this work for a finite time horizon $T > 0$ by replacing ∞ by $T > 0$.
- Note that this is precisely of the form $V_0 + G(\psi)$ that we saw in discrete time.

Itô Representation Theorem

Corollary 2

1. *Every (real-valued) local* $(P, \mathbb{F}^W)$*-martingale* L *is of the form*

$$L = L_0 + \int \gamma dW$$

   for some $\mathbb{R}^m$*-valued process* $\gamma \in L^2_{loc}(W)$
2. *Every local* $(P, \mathbb{F}^W)$*-martingale is continuous.*

- This provides a simple characterization of every local $(P, \mathbb{F}^W)$-martingale that can live in our probability space.
- It also gives an indication that our probability space needs to be richer if we want to be able to define say (compensated) Poisson process on that space.

Introduction to Black-Scholes Model

The Black-Scholes model is a continuous-time analogue of the symmetric multiplicative binomial model (also called the Cox-Ross-Rubinstein model).

This model can be described by the following set of SDEs

$$\begin{aligned} d\widetilde{S}^0_t &= \widetilde{S}^0_t r dt, & \widetilde{S}^0_0 &= 1 \\ d\widetilde{S}^1_t &= \widetilde{S}^1_t \mu dt + \widetilde{S}^1_t \sigma dW_t & \widetilde{S}^1_0 &= \widetilde{s}^1_0 \end{aligned}$$

for some constants $r, \mu \in \mathbb{R}$ and $\widetilde{s}^1_0, \sigma > 0$.

- $r \in \mathbb{R}$ corresponds to continuously compounded interest rate, so if we take $r' > -1$ as the simple interest rate from the CRR model, we obtain that
$$r = \log(1 + r').$$
- $\mu \in \mathbb{R}$ and $\sigma > 0$ correspond to the mean growth rate and volatility of the relative change in the price over an infinitesimal time step “dt”.
- An alternative interpretation is that $\left(\mu - \frac{1}{2}\sigma^2\right)$ and σ correspond to the mean and volatility of the logarithmic return of the stock over one unit of time, respectively.

Introduction to Black-Scholes Model

If SDEs admit explicit solutions at all, they can sometimes be found by applying Itô's formula to some $f : (x, t) \mapsto f(x, t)$ in $C^{2,2}$. For SDEs of the above form, $f(x) = \log(x)$ is a good guess. We obtain

$$\widetilde{S}_t^0 = \exp(rt)$$
$$\widetilde{S}_t^1 = \widetilde{s}_0^1 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

Defining logarithmic returns for $t \in \mathbb{N}$ as

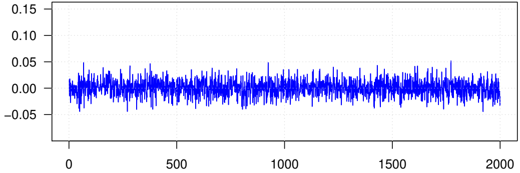
$$L_t = \log\left(\frac{\widetilde{S}_t^1}{\widetilde{S}_{t-1}^1}\right),$$

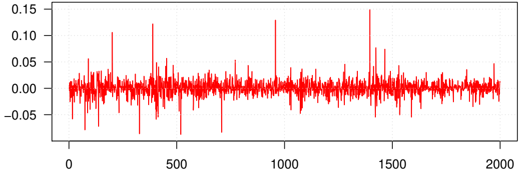
we have that L_t are i.i.d random variables with

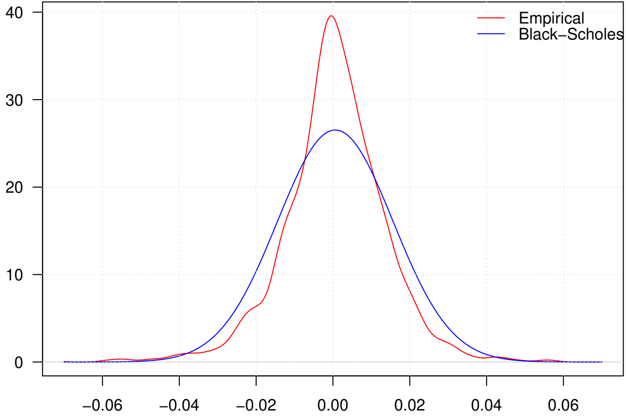
$$L_t \sim \mathcal{N}\left(\mu - \frac{1}{2}\sigma^2, \sigma^2\right),$$

so the mean is $\mu - \frac{1}{2}\sigma^2$ and standard deviation (volatility) is σ.

It is mentioned in the script that the Black-Scholes model as well as the CRR are too simple to be realistic. Why?






40
30
20
10
0
Empirical
Black–Scholes
–0.06
–0.04
–0.02
0.00
0.02
0.04
0.06

Thank you for your attention!