

Mathematical Foundations for Finance

Exercise sheet 13

Exercise 13.1 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. Let $M = (M_t)_{t \geq 0}$ be a local (P, \mathbb{F}) -martingale and $W = (W_t)_{t \geq 0}$ a (P, \mathbb{F}) -Brownian motion.

- (a) Let $H = (H_t)_{t \geq 0}$ be in $L^2(M)$. Compute $E \left[\int_0^T H_s dM_s \right]$ and $\text{Var} \left[\int_0^T H_s dM_s \right]$. How do the expressions look for $M := W$?
- (b) Let $H_s := \exp(-4s)$. Show that $\int_0^T H_s dW_s$ is in fact normally distributed. What are the mean and the variance of this normal distribution? How would the result change if $H : \mathbb{R} \rightarrow \mathbb{R}$ were an arbitrary (deterministic) continuous function?
Hint 1: Use the dominated convergence theorem for stochastic integrals from page 94 in the lecture notes.
Hint 2: If $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$, $X_n \rightarrow X$ in probability, $\mu_n \rightarrow \mu$ and $\sigma_n^2 \rightarrow \sigma^2 > 0$, then $X \sim \mathcal{N}(\mu, \sigma^2)$.
- (c) By coming up with a counterexample, show that the normality of $\int_0^T H_s dW_s$ from (b) does not hold for an arbitrary continuous $H \in L^2(W)$.

Solution 13.1

- (a) Since $H \in L^2(M)$, we know that $(H \cdot M) \in \mathcal{M}_0^2$, i.e. $(H \cdot M)$ is a square-integrable RCLL (P, \mathbb{F}) -martingale null at 0 with $\sup_{t \geq 0} E[(H \cdot M)_t^2] < \infty$. In particular, both $E[(H \cdot M)_T]$ and $\text{Var}[(H \cdot M)_T]$ are finite for all $T \geq 0$. We must therefore have that

$$E \left[\int_0^T H_s dM_s \right] = E \left[E \left[\int_0^T H_s dM_s \middle| \mathcal{F}_0 \right] \right] = E[(H \cdot M)_0] = 0.$$

For the variance, we compute

$$\text{Var} \left[\int_0^T H_s dM_s \right] = E \left[\left(\int_0^T H_s dM_s \right)^2 \right] = E \left[\int_0^T H_s^2 d[M]_s \right],$$

where the first equality uses that $E[(H \cdot M)_T] = 0$ and the second exploits the isometry property of the stochastic integral with respect to a local martingale.

In the particular case of Brownian motion, i.e. when $M = W$, no further simplification is needed for $E[(H \cdot W)_T]$. As for $\text{Var}[(H \cdot W)_T]$, we can compute

$$E \left[\int_0^T H_s^2 d[W]_s \right] = E \left[\int_0^T H_s^2 ds \right] = \int_0^T E[H_s^2] ds,$$

where the last equality uses Fubini's theorem.

- (b) Let us first discuss the hint. It is known that convergence in probability implies convergence in distribution, where sequence (X_n) of random variables converges in distribution (weakly) to a random variable X if

$$\lim_{n \rightarrow \infty} F_n(x) = F(x)$$

for every $x \in \mathbb{R}$ at which F is continuous. Here F_n is the distribution function of X_n and F the distribution function of X . But using the standard notation for the distribution function of $\mathcal{N}(0, 1)$, we have that

$$F_n(x) = \Phi\left(\frac{x - \mu_n}{\sigma_n}\right) \quad \text{and} \quad F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right),$$

and since Φ is a continuous function and $\sigma_n^2 \rightarrow \sigma > 0$, we have that

$$\lim_{n \rightarrow \infty} F_n(x) = \lim_{n \rightarrow \infty} \Phi\left(\frac{x - \mu_n}{\sigma_n}\right) = \Phi\left(\frac{x - \lim_{n \rightarrow \infty} \mu_n}{\lim_{n \rightarrow \infty} \sigma_n}\right) = \Phi\left(\frac{x - \mu}{\sigma}\right) = F(x)$$

for all $x \in \mathbb{R}$. We can thus conclude that (X_n) converges in distribution to a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$.

Now let $(\Pi_n)_{n \in \mathbb{N}}$ be a sequence of refining partitions of $[0, \infty)$ with $|\Pi_n| \rightarrow 0$ and define

$$H_s^n := \sum_{t_i \in \Pi_n} H_{t_i} \mathbb{1}_{(t_i, t_{i+1}]}(s) = \sum_{t_i \in \Pi_n} e^{-4t_i} \mathbb{1}_{(t_i, t_{i+1}]}(s). \quad (1)$$

Since we have that $H_s^n \rightarrow H_s$ pointwise for all $s \in \mathbb{R}_+$, then also $(H_s^n - H_s) \rightarrow 0$ pointwise for all $s \in \mathbb{R}_+$. But since all H^n and H are independent of $\omega \in \Omega$, we also have the pointwise convergence on $\bar{\Omega} := \Omega \times \mathbb{R}_+$. At the same time, we have for all $s \in \mathbb{R}_+$ that

$$|e^{-4s}| \leq 1 \implies |H_s^n - H_s| \leq 1, \quad (2)$$

so the dominated convergence theorem for stochastic integral from page 94 in the lecture notes says that $(H^n - H) \cdot W \rightarrow 0$ uniformly on compacts in probability (ucp), i.e.

$$\lim_{n \rightarrow \infty} P \left[\sup_{0 \leq s \leq T} |(H^n - H) \cdot W|_s > \epsilon \right] = 0 \quad \text{for all } \epsilon > 0.$$

But the above implies that

$$\lim_{n \rightarrow \infty} P \left[|(H^n - H) \cdot W|_T > \epsilon \right] = \lim_{n \rightarrow \infty} P \left[|(H^n \cdot W)_T - (H \cdot W)_T| > \epsilon \right] = 0 \quad \text{for all } \epsilon > 0,$$

so $(H^n \cdot W)_T \rightarrow (H \cdot W)_T$ in probability for all $T \geq 0$. Moreover,

$$(H^n \cdot W)_T = \sum_{t_i \in \Pi_n} H_{t_i} (W_{t_{i+1} \wedge T} - W_{t_i \wedge T}) \sim \mathcal{N}(m_n, s_n^2)$$

with

$$m_n = 0 \quad \text{and} \quad s_n^2 = \sum_{t_i \in \Pi_n} H_{t_i}^2 (t_{i+1} \wedge T - t_i \wedge T),$$

since Π_n is finite for all $n \in \mathbb{N}$ and the increments $W_{t_{i+1} \wedge T} - W_{t_i \wedge T}$ are independent and normally distributed with $\mathcal{N}(0, t_{i+1} \wedge T - t_i \wedge T)$. From the above, we see that

$$m_n \rightarrow 0 \quad \text{and} \quad s_n^2 \rightarrow \int_0^T H_s^2 ds = \int_0^T e^{-8s} ds = \frac{1}{8} (1 - e^{-8T}).$$

Hint 2 also gives us that the limiting distribution is additionally normal, so

$$\int_0^T e^{-4s} dW_s \sim \mathcal{N}\left(0, \frac{1}{8} (1 - e^{-8T})\right)$$

The only difference for an arbitrary continuous $H : \mathbb{R} \rightarrow \mathbb{R}$ is that the bound in (2) does not work. However for a fixed $T \geq 0$ we have that $H_s^n \mathbb{1}_{[0, T]}(s) \rightarrow H_s \mathbb{1}_{[0, T]}(s)$ pointwise for all

$s \in \mathbb{R}_+$ so $|H_s^n \mathbf{1}_{[0,T]}(s) - H_s \mathbf{1}_{[0,T]}(s)| \rightarrow 0$ pointwise for all $s \in \mathbb{R}_+$. Since both $H^n \mathbf{1}_{[0,T]}$ and $H \mathbf{1}_{[0,T]}$ are bounded, so is $|H^n \mathbf{1}_{[0,T]} - H \mathbf{1}_{[0,T]}|$. Additionally, $H^n \mathbf{1}_{[0,T]}$ and $H \mathbf{1}_{[0,T]}$ are also left-continuous and therefore predictable, so the dominated convergence theorem is indeed applicable and we can proceed as before. We obtain that

$$\int_0^T H_s^n \mathbf{1}_{[0,T]}(s) dW_s \rightarrow \int_0^T H_s \mathbf{1}_{[0,T]}(s) dW_s = \int_0^T H_s dW_s \quad \text{in probability.}$$

- (c) We have seen in the lecture (see for instance the example on page 102 of the lecture notes) that

$$\int_0^T W_s dW_s = \frac{1}{2} W_T^2 - \frac{1}{2} T.$$

Since W_T^2 only takes positive values, it is clear that $\frac{1}{2} W_T^2 - \frac{1}{2} T$ is bounded from below for every $T \geq 0$, so it cannot be normally distributed. More specifically,

$$\frac{1}{2} W_T^2 - \frac{1}{2} T \stackrel{d}{=} \frac{1}{2} T^2 W_1^2 - \frac{1}{2} T,$$

and since $W_1 \sim \mathcal{N}(0, 1)$, we have that $W_1^2 \sim \chi_1^2$, so the distribution of $\int_0^T W_s dW_s$ is just a simple affine transformation of a chi-squared distribution with one degree of freedom.

Exercise 13.2 Let $T > 0$ denote a fixed time horizon and $W = (W_t)_{t \in [0,T]}$ a Brownian motion on some probability space (Ω, \mathcal{F}, P) . Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ be the filtration generated by W and augmented by the P -nullsets in $\sigma(W_s; s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\tilde{S}^0 = (\tilde{S}_t^0)_{t \in [0,T]}$ and the undiscounted stock price process $\tilde{S}^1 = (\tilde{S}_t^1)_{t \in [0,T]}$ are given by

$$d\tilde{S}_t^0 = \tilde{S}_t^0 r dt \quad \text{and} \quad d\tilde{S}_t^1 = \tilde{S}_t^1 (\mu dt + \sigma dW_t), \tag{3}$$

where $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\tilde{S}_0^0 = 1$ and $\tilde{S}_0^1 > 0$ are deterministic.

- (a) Prove using Itô's formula that the discounted stock price process $S^1 = \tilde{S}^1 / \tilde{S}^0$ solves

$$dS_t^1 = S_t^1 ((\mu - r) dt + \sigma dW_t). \tag{4}$$

- (b) Prove using Itô's formula that

$$S^1 = \left(S_0^1 \exp \left(\sigma W_t + \left(\mu - r - \frac{1}{2} \sigma^2 \right) t \right) \right)_{t \in [0,T]},$$

i.e. show that the process $(S_0^1 \exp(\sigma W_t + (\mu - r - \frac{1}{2} \sigma^2) t))_{t \in [0,T]}$ solves (4).

- (c) Let $L^\lambda := -\lambda W$ and $Z^\lambda := \mathcal{E}(L^\lambda)$. Prove that the process $W^\lambda := (W_t + \lambda t)_{t \in [0,T]}$ is a Brownian motion under the measure Q_λ given by $\frac{dQ_\lambda}{dP} := Z_T^\lambda$.

- (d) Prove that for the right choice of λ , the discounted stock price process S^1 is a Q_λ -martingale. *Hint: Rewrite $\sigma W_t + (\mu - r - \frac{1}{2} \sigma^2) t$ as function of $W_t^\lambda, t, \sigma, \mu$, and r .*

Solution 13.2

- (a) Using that $\tilde{S}^0 > 0$ P -a.s., we can apply Itô's formula to the C^2 -function $f : \mathbb{R}_{++} \times \mathbb{R}_{++} \rightarrow \mathbb{R}$ given by $f(x, y) := x/y$. Computing the required derivatives, we obtain

$$\frac{\partial f}{\partial x}(x, y) = \frac{1}{y}, \quad \frac{\partial f}{\partial y}(x, y) = -\frac{x}{y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(x, y) = 0,$$

and, moreover, since \tilde{S}^0 is of finite variation,

$$\langle \tilde{S}^1, \tilde{S}^0 \rangle = 0 \quad \text{and} \quad \langle \tilde{S}^0, \tilde{S}^0 \rangle = 0.$$

By Itô's formula, we obtain

$$\begin{aligned} S_t^1 &= \frac{\tilde{S}_t^1}{\tilde{S}_t^0} = S_0^1 + \int_0^t \frac{1}{\tilde{S}_s^0} d\tilde{S}_s^1 + \int_0^t \frac{-\tilde{S}_s^1}{(\tilde{S}_s^0)^2} d\tilde{S}_s^0 + 0 \\ &= S_0^1 + \int_0^t \frac{1}{\tilde{S}_s^0} \left(\tilde{S}_s^1 (\mu ds + \sigma dW_s) \right) + \int_0^t \frac{-\tilde{S}_s^1}{(\tilde{S}_s^0)^2} \left(\tilde{S}_s^0 r ds \right) \\ &= S_0^1 + \int_0^t S_s^1 ((\mu - r) ds + \sigma dW_s), \end{aligned}$$

or, written equivalently in the differential notation,

$$dS_t^1 = S_t^1 ((\mu - r) dt + \sigma dW_t).$$

- (b) Consider the C^2 -function $f : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ given by $f(w, t) := S_0^1 \exp(\sigma w + (\mu - r - \sigma^2/2)t)$. Computing the different derivatives, we obtain

$$\frac{\partial f}{\partial w}(w, t) = \sigma f(w, t), \quad \frac{\partial f}{\partial t}(w, t) = (\mu - r - \sigma^2/2)f(w, t), \quad \text{and} \quad \frac{\partial^2 f}{\partial w^2}(w, t) = \sigma^2 f(w, t),$$

and moreover, since $(t)_{t \in [0, T]}$ is of finite variation,

$$\langle W \rangle_t = t, \quad \langle t, W \rangle = \langle W, t \rangle = 0, \quad \text{and} \quad \langle t \rangle = 0.$$

By Itô's formula, we can then compute

$$\begin{aligned} f(W_t, t) &= S_0^1 + \int_0^t \sigma f(W_s, s) dW_s + \int_0^t \left(\mu - r - \frac{\sigma^2}{2} \right) f(W_s, s) ds + \frac{1}{2} \int_0^t \sigma^2 f(W_s, s) ds \\ &= S_0^1 + \int_0^t f(W_s, s) (\sigma dW_s + (\mu - r) ds), \end{aligned}$$

and thus conclude that $f(W_t, t) = S_0^1 \exp(\sigma W_t + (\mu - r - \sigma^2/2)t)$ solves the SDE.

- (c) Since $\lambda t = \int_0^t \lambda ds$, we can write $W_t^\lambda = W_t + \int_0^t \lambda ds$. Using that $L_t = -\lambda W_t = \int_0^t -\lambda dW_s$, we can then directly conclude by Girsanov's theorem (Theorem VI.2.3 in the lecture notes) that W^λ is a Q_λ -Brownian motion.
- (d) Note that we can write

$$\sigma W_t + \left(\mu - r - \frac{1}{2} \sigma^2 \right) t = \sigma (W_t^\lambda - \lambda t) + \left(\mu - r - \frac{1}{2} \sigma^2 \right) t = \sigma W_t^\lambda + \left(\mu - r - \frac{1}{2} \sigma^2 - \sigma \lambda \right) t.$$

Since by (c) W^λ is a Brownian motion under Q_λ , we can deduce from Proposition IV.2.2 in the lecture notes that the process

$$\left(S_0^1 \exp \left(\sigma W_t + \left(\mu - r - \frac{1}{2} \sigma^2 \right) t \right) \right)_{t \in [0, T]} \quad (5)$$

is a Q_λ -martingale if and only if $\mu - r - \frac{1}{2} \sigma^2 - \sigma \lambda = -\frac{1}{2} \sigma^2$. Solving this equation, we can conclude by (b), since S^1 coincides with the process in (5), that S^1 is a Q^λ martingale if and only if $\lambda = \frac{\mu - r}{\sigma}$.

Exercise 13.3 Let $T > 0$ denote a fixed time horizon and let $W = (W_t)_{t \in [0, T]}$ be a Brownian motion on some probability space (Ω, \mathcal{F}, P) . Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ be the filtration generated by W and augmented by the P -nullsets in $\sigma(W_s; 0 \leq s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\tilde{S}^0 = (\tilde{S}_t^0)_{t \in [0, T]}$ and the undiscounted stock price process $\tilde{S}^1 = (\tilde{S}_t^1)_{t \in [0, T]}$ are given by

$$\frac{d\tilde{S}_t^0}{\tilde{S}_t^0} = r dt \quad \text{and} \quad \frac{d\tilde{S}_t^1}{\tilde{S}_t^1} = \mu dt + \sigma dW_t,$$

with $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\tilde{S}_0^0 = 1$ and $\tilde{S}_0^1 > 0$ deterministic. Using the notation of the previous exercise, denote $Q^* := Q_{\lambda^*}$, where λ^* is the unique value of λ making Q_λ an equivalent martingale measure for $S^1 := \tilde{S}^1 / \tilde{S}^0$.

Hint: If you did not find λ^ in Exercise 13.2 (d), you can use that $\lambda^* = \frac{\mu - r}{\sigma}$.*

- (a) Hedge the *square option*, i.e., find a self-financing strategy $\varphi \hat{=} (V_0, \vartheta)$ such that

$$V_0 + \int_0^T \vartheta_u dS_u^1 = \frac{(\tilde{S}_T^1)^2}{\tilde{S}_T^0}.$$

Hint: Look for a representation result under Q^ , not under P .*

- (b) Hedge the *inverted option*, i.e., find a self-financing strategy $\varphi \hat{=} (\bar{V}_0, \bar{\vartheta})$ such that

$$\bar{V}_0 + \int_0^T \bar{\vartheta}_u dS_u^1 = \frac{1}{\tilde{S}_T^0 \tilde{S}_T^1}.$$

Solution 13.3 Set $\lambda^* := \frac{\mu - r}{\sigma}$. We know from the previous exercise that $W_t^* := W_t + \lambda^* t$, $t \in [0, T]$ is a Brownian motion under the equivalent martingale measure Q^* . Moreover, the discounted stock price process $S^1 = \frac{\tilde{S}^1}{\tilde{S}^0}$ is a Q^* -martingale and is explicitly given by

$$S_t = S_0^1 e^{\sigma W_t^* - \frac{1}{2} \sigma^2 t}.$$

The *discounted* arbitrage-free value at time t of any *discounted* payoff $H \in L_+^1(\mathcal{F}_T, Q^*)$ is given by

$$V_t^* = E_{Q^*} [H | \mathcal{F}_t].$$

- (a) We use the discussion from page 114 of the lecture notes to conclude that $V_t^* = E_{Q^*} [H | \mathcal{F}_t]$ may be represented as a stochastic integral of the form

$$V_t^* = E_{Q^*} [H] + \int_0^t \varphi_s dS_s^1, \quad 0 \leq t \leq T.$$

In this case, we compute

$$\begin{aligned} V_t^* &= e^{-rT} E_{Q^*} \left[(\tilde{S}_T^1)^2 \middle| \mathcal{F}_t \right] = e^{-rT} e^{2Tr} E_{Q^*} \left[(S_T^1)^2 \middle| \mathcal{F}_t \right] \\ &= e^{rT} (S_t^1)^2 E_{Q^*} \left[e^{2\sigma(W_T^* - W_t^*) - \sigma^2(T-t)} \middle| \mathcal{F}_t \right] \\ &= e^{(r+\sigma^2)T - \sigma^2 t} (S_t^1)^2 =: v(t, S_t^1). \end{aligned} \tag{6}$$

We apply Itô's formula to v and obtain

$$v(t, S_t^1) = v(0, S_0^1) + \int_0^t \frac{\partial}{\partial x} v(t, S_t^1) dS_t^1 + \text{continuous FV process.}$$

Since the left-hand side and the stochastic integral on the right-hand side are local (Q^*, \mathbb{F}) -martingales, the “continuous FV process” is a local (Q^*, \mathbb{F}) -martingale as well and since it apparently is null at 0, it must be identically equal to 0 (see the second hint from Exercise 11.1 (d)). We thus immediately obtain that

$$\vartheta_t = \frac{\partial}{\partial x} v(t, S_t^1) = 2e^{(r+\sigma^2)T-\sigma^2 t} S_t^1 = e^{(r+\sigma^2)(T-t)} \tilde{S}_t^1.$$

For $v(0, S_0^1)$, we have that $v(0, S_0^1) = e^{(r+\sigma^2)T} (S_0^1)^2$.

(b) We proceed analogously to (a). Here, we obtain

$$\begin{aligned} \bar{V}_t^* &= e^{-rT} E_{Q^*} \left[\frac{1}{\tilde{S}_T^1} \middle| \mathcal{F}_t \right] = e^{-2rT} \frac{1}{S_t^1} E_{Q^*} \left[\frac{S_t^1}{S_T^1} \middle| \mathcal{F}_t \right] \\ &= e^{-2rT} \frac{1}{S_t^1} E_{Q^*} \left[e^{-\sigma(W_T^* - W_t^*) + \frac{1}{2}\sigma^2(T-t)} \middle| \mathcal{F}_t \right] \\ &= e^{(\sigma^2 - 2r)T - \sigma^2 t} \frac{1}{S_t^1} \\ &=: \bar{v}(t, S_t^1). \end{aligned}$$

We conclude again in the same way as in (a) that

$$\begin{aligned} \bar{\vartheta}_t &= \frac{\partial}{\partial x} \bar{v}(t, S_t^1) = -e^{(\sigma^2 - 2r)T - \sigma^2 t} \frac{1}{(S_t^1)^2} = -e^{(\sigma^2 - 2r)(T-t)} \frac{1}{(\tilde{S}_t^1)^2}, \\ \bar{V}_0^* &= v(0, S_0^1) = e^{(\sigma^2 - 2r)T} \frac{1}{S_0^1} = e^{(\sigma^2 - 2r)T} \frac{1}{\tilde{S}_0^1}. \end{aligned}$$