ETH Zürich, Fall 2018
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Mathematical Foundations for Finance

Exercise sheet 14

This exercise sheet will not be corrected. Please do not hand in.

Exercise 14.1 We will now use the techniques that we have developed in the course in a slightly different setting. Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ on a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let W^1 and W^2 be two $(P, \mathbb{F})$-Brownian motions with (constant) correlation $\rho \in (-1, 1)$ and let the dynamics of $\widetilde{S}^0$ and $\widetilde{S}^1$ be described by the SDEs

$$\begin{aligned} d\widetilde{S}^0_t &= \widetilde{S}^0_t r_t dt, \\ dr_t &= \theta(\alpha - r_t)dt + \eta dW^1_t, \\ d\widetilde{S}^1_t &= \widetilde{S}^1_t \left(r_t dt + \sigma dW^2_t \right), \end{aligned}$$

where $\sigma, \eta > 0$ and $\theta, \alpha \in \mathbb{R}$ as well as $\widetilde{S}^0_0 = 1$, $\widetilde{S}^1_0 > 0$ and $r_0 \in \mathbb{R}$ are all constant. This is the Black–Scholes model with stochastic interest rate.

(a) By applying Itô's formula to some function $f \in C^2$ and the continuous semimartingale $\widetilde{S}^0$, show that the solution to the first SDE is given by

$$\widetilde{S}^0_t = \exp\left(\int_0^t r_s ds \right).$$

(b) By applying Itô's formula to the function $f(x, t) = xe^{\theta t}$ and the continuous semimartingale $(r_t, t)_{t\geq 0}$, show that the solution to the second SDE is given by

$$r_t = r_0 e^{-\theta t} + \alpha(1 - e^{-\theta t}) + \eta e^{-\theta t} \int_0^t e^{\theta s} dW_s.$$

The solution to this SDE is called *Ornstein–Uhlenbeck process* and is an important process when it comes to interest rate modeling.

(c) Show that the discounted price processes $S^0 := \widetilde{S}^0/\widetilde{S}^0$ and $S^1 := \widetilde{S}^1/\widetilde{S}^0$ are $(P, \mathbb{F})$-martingales, i.e. the market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free and we can use P as our pricing measure.

Exercise 14.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion with respect to a probability measure P and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. We know from the lecture that if $H \in L^2_{\mathrm{loc}}(W)$, then $\int H dW$ is a local $(P, \mathbb{F})$-martingale. The purpose of this exercise is to study when $\int H dW$ is a even a true $(P, \mathbb{F})$-martingale. Show that $\int H dW$ is a $(P, \mathbb{F})$-martingale on $[0, \infty)$ if

(a) $\int H dW$ is a $(P, \mathbb{F})$-martingale on $[0, T]$ for every $0 \leq T < \infty$.

(b) $\int H dW$ has a majorant in $L^1(P)$ on $[0, T]$, i.e.

$$\left| \int_0^t H_s dW_s \right| \leq X \quad \text{for all } t \in [0, T],$$

where $X \in L^1(P)$.

(c) $H \in L^2(W^T)$, i.e. $E\left[\int_0^T H_s^2 ds\right] < \infty$.

Exercise 14.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a Brownian motion $W = (W_t)_{t\in[0,T]}$. Let $\mathbb{F} := \mathbb{F}^W$ be the (augmented) filtration generated by W. Consider the discounted price process $S = (S_t)_{t\in[0,T]}$ with dynamics

$$dS_t = \sigma(t, S_t)dW_t, \quad S_0 > 0,$$

where $\sigma : [0,T] \times \mathbb{R} \to (0,\infty)$ is a continuous and bounded function. One can show that S is well defined and P is the unique EMM for S. Let $h : \mathbb{R} \to [0,\infty)$ be a fixed continuous and bounded function. We consider the partial differential equation (PDE)

$$\begin{cases} \frac{\partial}{\partial t}v(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2}v(t,x) = 0, & x \in (0,T)\times\mathbb{R}, \\ v(T,x) = h(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

Suppose that there exists a $C^{1,2}$ solution $v : [0,T]\times\mathbb{R} \to \mathbb{R}$ of (1) with the additional property that

$$\left|\sigma(t,x)\frac{\partial}{\partial x}v(t,x)\right| \le C(1+|x|), \quad (t,x) \in [0,T]\times\mathbb{R},$$

for some constant $C > 0$. Show that $V_t^* := v(t, S_t)$, $t \in [0,T]$, is the price at time t of the discounted European contingent claim $h(S_T)$.