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Mathematical Foundations for Finance

Exercise sheet 14

Exercise 14.1 We will now use the techniques that we have developed in the course in a slightly different setting. Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ on a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let W^1 and W^2 be two $(P, \mathbb{F})$-Brownian motions with (constant) correlation $\rho \in (-1, 1)$ and let the dynamics of $\widetilde{S}^0$ and $\widetilde{S}^1$ be described by the SDEs

$$\begin{aligned} d\widetilde{S}^0_t &= \widetilde{S}^0_t r_t dt, \\ dr_t &= \theta(\alpha - r_t)dt + \eta dW^1_t, \\ d\widetilde{S}^1_t &= \widetilde{S}^1_t \left(r_t dt + \sigma dW^2_t\right), \end{aligned}$$

where $\sigma, \eta > 0$ and $\theta, \alpha \in \mathbb{R}$ as well as $\widetilde{S}^0_0 = 1$, $\widetilde{S}^1_0 > 0$ and $r_0 \in \mathbb{R}$ are all constant. This is the Black–Scholes model with stochastic interest rate.

(a) By applying Itô's formula to some function $f \in C^2$ and the continuous semimartingale $\widetilde{S}^0$, show that the solution to the first SDE is given by

$$\widetilde{S}^0_t = \exp\left(\int_0^t r_s ds\right).$$

(b) By applying Itô's formula to the function $f(x, t) = xe^{\theta t}$ and the continuous semimartingale $(r_t, t)_{t\geq 0}$, show that the solution to the second SDE is given by

$$r_t = r_0 e^{-\theta t} + \alpha(1 - e^{-\theta t}) + \eta e^{-\theta t} \int_0^t e^{\theta s} dW_s.$$

The solution to this SDE is called *Ornstein–Uhlenbeck process* and is an important process when it comes to interest rate modeling.

(c) Show that the discounted price processes $S^0 := \widetilde{S}^0/\widetilde{S}^0$ and $S^1 := \widetilde{S}^1/\widetilde{S}^0$ are $(P, \mathbb{F})$-martingales, i.e. the market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free and we can use P as our pricing measure.

Solution 14.1

(a) Due to the similarity with the ordinary differential equation $\frac{y'}{y} = g \iff \log(y)' = g$, whose solution is given by $y(t) = C \exp\left(\int g(t)dt\right)$, one might try to apply Itô's formula to the function $f(x) = \log(x)$ and the positive continuous semimartingale $\widetilde{S}^0$. This yields

$$\begin{aligned} \log(\widetilde{S}^0_t) &= \log(\widetilde{S}^0_0) + \int_0^t \frac{1}{\widetilde{S}^0_s} d\widetilde{S}^0_s - \frac{1}{2}\int_0^t \frac{1}{(\widetilde{S}^0_t)^2} d[\widetilde{S}^0]_s \\ &= \int_0^t \frac{1}{\widetilde{S}^0_s}\widetilde{S}^0_s r_s ds = \int_0^t r_s ds, \end{aligned}$$

where we have used that $\widetilde{S}^0$ is of finite variation and therefore

$$[\widetilde{S}^0]_t = \left[\int \widetilde{S}^0 r ds\right]_t = \int_0^s (\widetilde{S}^0_s)^2 r_s^2 d[s]_s = 0,$$

so indeed

$$\widetilde{S}_t^0 = \exp\left(\int_0^t r_s ds\right).$$

(b) Since we have that $[t] = 0$ and $[W, t] = 0$, Itô's formula gives

$$f(r_t, t) = f(r_0, 0) + \int_0^t \frac{\partial f}{\partial x}(r_s, s) dr_s + \int_0^t \frac{\partial f}{\partial t}(r_s, s) ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(r_s, s) d[r]_s.$$

The required derivatives read

$$\frac{\partial f}{\partial x}(x,t) = e^{\theta t}, \quad \frac{\partial f}{\partial t}(x,t) = \theta x e^{\theta t}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(x,t) = 0.$$

Using the above and the fact that $d[r]_t = \eta^2 d[W^1]_t = \eta^2 dt$, we obtain

$$\begin{aligned} r_t e^{\theta t} &= r_0 + \int_0^t e^{\theta s}\theta\alpha ds - \int_0^t e^{\theta s}\theta r_s ds + \int_0^t e^{\theta s}\eta dW_s^1 + \int_0^t e^{\theta s}\theta r_s ds \\ &= r_0 + \theta\alpha \int_0^t e^{\theta s} ds + \eta \int_0^t e^{\theta s} dW_s^1 \\ &= r_0 + \alpha(e^{\theta t} - 1) + \eta \int_0^t e^{\theta s} dW_s^1. \end{aligned}$$

Multiplying both sides of the equation by $e^{-\theta t}$ gives the desired result.

(c) Since $S^0 \equiv 1$, it is clearly a $(P, \mathbb{F})$-martingale. As for $S^1 = \widetilde{S}^1/\widetilde{S}^0$, we do this by applying Itô's formula to the C^2 function $f(x,y) = \frac{x}{y}$ and the semimartingale $(\widetilde{S}_t^1, \widetilde{S}_t^0)_{t\geq 0}$. Since we have seen in (a) that $\widetilde{S}^0$ is of finite variation and $[\widetilde{S}^0] = 0$, we also have that $[\widetilde{S}^1, \widetilde{S}^0] = [\widetilde{S}^0, \widetilde{S}^1] = 0$. Itô's formula therefore gives

$$S_t^1 = S_0^1 + \int_0^t \frac{\partial f}{\partial x}(\widetilde{S}_s^1, \widetilde{S}_s^0) d\widetilde{S}_s^1 + \int_0^t \frac{\partial f}{\partial y}(\widetilde{S}_s^1, \widetilde{S}_s^0) d\widetilde{S}_s^0 + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(\widetilde{S}_s^1, \widetilde{S}_s^0) d[\widetilde{S}^1]_s.$$

A direct computation of the required derivatives yields

$$\frac{\partial}{\partial x} f(x,y) = \frac{1}{y}, \quad \frac{\partial}{\partial y} f(x,y) = -\frac{x}{y^2}, \quad \frac{\partial^2}{\partial x^2} f(x,y) = 0, \tag{1}$$

giving us in turn that

$$S_t^1 = S_0^1 + \int_0^t S_s^1 r_s ds + \sigma \int_0^t S_s^1 dW_s^2 - \int_0^t S_s^1 r_s ds = S_0^1 + \sigma \int_0^t S_s^1 dW_s^2. \tag{2}$$

Note that we can rewrite (2) in the differential form as $dS_t^1 = \sigma S_t^1 dW_t^2$. In order to show that S^1 is in fact a true $(P, \mathbb{F})$-martingale, we apply Itô's formula to the C^2 function $f(x) = \log(x)$ and the local $(P, \mathbb{F})$-martingale (thus a $(P, \mathbb{F})$-semimartingale) S^1. Similarly to (a), this yields

$$\begin{aligned} \log(S_t^1) &= \log(S_0^1) + \int_0^t \sigma \frac{1}{S_s^1} S_s^1 dW_s^2 - \frac{1}{2}\int_0^t \sigma^2 \frac{1}{(S_s^1)^2}(S_s^1)^2 ds \\ &= \log(S_0^1) + \sigma W_t^2 - \frac{1}{2}\sigma^2 t, \end{aligned}$$

or, equivalently, $S_t^1 = S_0^1 \exp\left(\sigma W_t^2 - \frac{1}{2}\sigma^2 t\right)$. Alternatively, $dS_t^1 = S_t^1 \sigma dW_t^2$ gives directly that

$$S_t^1 = S_0^1 \mathcal{E}(\sigma W^2)_t = S_0^1 \exp\left(\sigma W_t^2 - \frac{1}{2}\sigma^2 t\right).$$

But as we have seen in Proposition IV.2.2 in the lecture notes, the exponential term is a $(P, \mathbb{F})$-martingale, and since S_0^1 is just a constant, S^1 is also a $(P, \mathbb{F})$-martingale.

Exercise 14.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion with respect to a probability measure P and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. We know from the lecture that if $H \in L^2_{\text{loc}}(W)$, then $\int HdW$ is a local $(P, \mathbb{F})$-martingale. The purpose of this exercise is to study when $\int HdW$ is a even a true $(P, \mathbb{F})$-martingale. Show that $\int HdW$ is a $(P, \mathbb{F})$-martingale on $[0, \infty)$ if

(a) $\int HdW$ is a $(P, \mathbb{F})$-martingale on $[0, T]$ for every $0 \leq T < \infty$.

(b) $\int HdW$ has a majorant in $L^1(P)$ on $[0, T]$, i.e.

$$\left| \int_0^t H_s dW_s \right| \leq X \quad \text{for all } t \in [0, T],$$

where $X \in L^1(P)$.

(c) $H \in L^2(W^T)$, i.e. $E\left[\int_0^T H_s^2 ds\right] < \infty$.

Solution 14.2

(a) Let $I_t := \int_0^t H_u dW_u$, $t \geq 0$. Since we have

$$[0, \infty) = \bigcup_{T \geq 0} [0, T],$$

I_t is integrable for all $t \in [0, T]$ and all $T \in [0, \infty)$ if and only if I_t is integrable for every $t \in [0, \infty)$. Likewise, the martingale property $E[I_t \,|\, \mathcal{F}_s] = I_s$ P-a.s. holds for all $s \leq t$ in $[0, T]$ and all $T \in [0, \infty)$ if and only if it holds for every $s \leq t$ in $[0, \infty)$.

(b) By (a) it is sufficient to consider $I_t := \int_0^t H_u dW_u$ for $t \in [0, T]$ with a $T > 0$ fixed. We know that the process $\int HdW$ is a local $(P, \mathbb{F})$-martingale. Let $(\sigma_n)_{n\in\mathbb{N}}$ be a localizing sequence for $\int HdW$ on $\Omega \times [0, \infty)$. Then $\tau_n := \sigma_n \wedge T$ is a localizing sequence for $I = (I_t)_{t\in[0,T]}$ on $\Omega \times [0, T]$. Moreover, by assumption, there exists a random variable $X \in L^1(P)$ such that

$$|I_t| = \left| \int_0^t H_u dW_u \right| \leq X \quad P\text{-a.s. for all } t \in [0, T].$$

Let $s \leq t$. We have $\tau_n \uparrow T$ P-a.s., so $T_t^{\tau_n} \to I_t$ and $I_s^{\tau_n} \to I_s$, and the dominated convergence for conditional expectations therefore gives

$$E[I_t \,|\, \mathcal{F}_s] \, E\left[\lim_{n\to\infty} I_t^{\tau_n} \,\middle|\, \mathcal{F}_s\right] = \lim_{n\to\infty} E[I_t^{\tau_n} \,|\, \mathcal{F}_s] = \lim_{n\to\infty} I_s^{\tau_n} = I_s \quad P\text{-a.s.},$$

i.e., the process I is a $(P, \mathbb{F})$-martingale, and by (a), so is $\int HdW$.

(c) Fix $T \geq 0$. Since $(W_t^2 - t)_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale, the process

$$\left(W_{t\wedge T}^2 - (t \wedge T)\right)_{t\geq 0}$$

is a $(P, \mathbb{F})$-martingale as well by the stopping theorem from page 69 in the lecture notes. Moreover the process $(t \wedge T)_{t\geq 0}$ is also adapted to $\mathbb{F}$, increasing, null at 0 with $\Delta(t \wedge T) = 0 = (\Delta W_{t\wedge T})^2$, so, as a result

$$\left[W^T\right]_t = t \wedge T$$

for all $t \geq 0$, hence

$$E\left[\int_0^\infty H_s^2 d\left[W^T\right]_s\right] = E\left[\int_0^\infty H_s^2 d(s \wedge T)\right] = E\left[\int_0^T H_s^2 ds\right] < \infty$$

by our assumption that $H \in L^2(W^T)$. Since it is clear that $W^T \in \mathcal{M}_0^2$, this directly implies that $\int H dW^T \in \mathcal{M}_0^2$ (see the section "(local) martingale property" at page 86 of the lecture notes). Moreover, since

$$\int H_s dW_s^T = \left(\int H_s dW_s \right)^T$$

(see section "behavior under stopping" at page 88), we get that $\left(\int H_s dW_s \right)^T \in \mathcal{M}_0^2$ as well, and this in particular gives us that $\left(\int_0^t H_s dW_s \right)_{0 \le t \le T}$ is a square-integrable $(P, \mathbb{F})$-martingale on $[0, T]$. Since T was arbitrary, $\int H dW$ is a $(P, \mathbb{F})$-martingale by (a).

Exercise 14.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a Brownian motion $W = (W_t)_{t \in [0,T]}$. Let $\mathbb{F} := \mathbb{F}^W$ be the (augmented) filtration generated by W. Consider the discounted price process $S = (S_t)_{t \in [0,T]}$ with dynamics

$$dS_t = \sigma(t, S_t) dW_t, \quad S_0 > 0,$$

where $\sigma : [0, T] \times \mathbb{R} \to (0, \infty)$ is a continuous and bounded function. One can show that S is well defined and P is the unique EMM for S. Let $h : \mathbb{R} \to [0, \infty)$ be a fixed continuous and bounded function. We consider the partial differential equation (PDE)

$$\begin{cases} \frac{\partial}{\partial t} v(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2}{\partial x^2} v(t, x) = 0, & x \in (0, T) \times \mathbb{R}, \\ v(T, x) = h(x), & x \in \mathbb{R}. \end{cases} \tag{3}$$

Suppose that there exists a $C^{1,2}$ solution $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ of (3) with the additional property that

$$\left| \sigma(t, x) \frac{\partial}{\partial x} v(t, x) \right| \le C(1 + |x|), \quad (t, x) \in [0, T] \times \mathbb{R},$$

for some constant $C > 0$. Show that $V_t^* := v(t, S_t)$, $t \in [0, T]$, is the price at time t of the discounted European contingent claim $h(S_T)$.

Solution 14.3 By Itô's formula, we have

$$\begin{aligned} V_t^* = v(t, S_t) &= v(0, S_0) + \int_0^t \frac{\partial}{\partial x} v(t, S_t) dS_t + \int_0^t \left(\frac{\partial}{\partial t} v(t, S_t) + \frac{1}{2} \sigma^2(t, S_t) \frac{\partial^2}{\partial x^2} v(t, S_t) \right) dt \\ &= v(0, S_0) + \int_0^t \frac{\partial}{\partial x} v(t, S_t) dS_t, \end{aligned}$$

where the "dt" integral disappears by the assumption that v solves the given PDE. We now claim that the stochastic integral $\int \frac{\partial}{\partial x} v(t, S_t) dS_t$ is in fact a square-integrable $(P, \mathbb{F})$-martingale. To that end, we recall that since σ is a bounded function and $\sup_{0 \le t \le T} E\left[W_t^2\right] \le T$, i.e. $W^T \in \mathcal{M}_0^2$, S^T is a $(P, \mathbb{F})$-martingale in $\mathcal{M}_0^2$ by Proposition V.1.3 in the lecture notes. Hence $\sup_{0 \le u \le T} |S_u| \in L^2(P)$. Moreover,

$$\begin{aligned} \left\| \frac{\partial}{\partial x} v(t, S_t) \right\|_{L^2(S)}^2 &= E\left[\int_0^T \frac{\partial}{\partial x} v(t, S_t) dS_t \right] = E\left[\int_0^T \left(\frac{\partial}{\partial x} v(t, S_t) \right)^2 d[S]_t \right] \\ &= E\left[\int_0^T \left(\sigma(t, S_t) \frac{\partial}{\partial x} v(t, S_t) \right)^2 dt \right] \\ &\le E\left[C^2 T \left(1 + \sup_{0 \le t \le T} |S_t| \right)^2 \right] < \infty, \end{aligned}$$

due to the assumption that

$$\left| \sigma(t, x) \frac{\partial}{\partial x} v(t, x) \right| \le C(1 + |x|), \quad (t, x) \in [0, T] \times \mathbb{R},$$

and the fact that $\sup_{0\le u\le T}|S_u| \in L^2(P)$. Since $\frac{\partial}{\partial x}v(t,S)$ is continuous and $\mathbb{F}$-adapted, therefore predictable, it means that $\frac{\partial}{\partial x}v(t,S) \in L^2(S)$ and the stochastic integral $\int \frac{\partial}{\partial x}v(t,S_t)dS_t$ is in fact a $(P,\mathbb{F})$-martingale on $[0,T]$. We conclude that V^* is a true $(P,\mathbb{F})$-martingale with $V_T^* = h(S_T)$. So

$$v(t,S_t) = V_t^* = E_P\left[v(T,S_T)\,|\,\mathcal{F}_t\right] = E_P\left[h(S_T)\,|\,\mathcal{F}_t\right]$$

is the price at time t of the claim $h(S_T)$.