ETH zürich

Mathematical Foundations for Finance

Exercise 2

Martin Stefanik
ETH Zurich

Notation

One should get familiar with the following notation very quickly.

$\widetilde{S}^0$	the bank account process
$\widetilde{S}^k$	the price process of the k-th risky asset $(k = 1, \ldots, d)$
$\widetilde{S}^k_n$	the price of the k-th asset at time n $(k = 0, 1, \ldots, d)$
S^k	the discounted price process of k-th asset $(k = 0, 1, \ldots, d)$
$\varphi = (\varphi^0, \vartheta)$	an arbitrary trading strategy $(\varphi^0 \in \mathbb{R}, \vartheta \in \mathbb{R}^d)$
$V(\varphi)$	the discounted value process of the trading strategy φ
$G(\varphi)$	the discounted gains process of the trading strategy φ
$C(\varphi)$	the discounted cost process of the trading strategy φ

Trading strategy is one of the most basic concepts in mathematical finance. The definition is just a formalization of what one typically imagines under a rule-based strategy for allocating money into financial markets.

Definition 1 (Trading strategy)

A *trading strategy* is an $\mathbb{R}^{d+1}$-valued stochastic process $\varphi = (\varphi^0, \vartheta)$, where $\varphi^0 = (\varphi_k^0)_{k=0,1,\dots,T}$ is real-valued and adapted and $\vartheta = (\vartheta_k)_{k=0,1,\dots,T}$ with $\vartheta_0 = 0$ is $\mathbb{R}^d$-valued and predictable.

Definition 2 (Discounted value process)

The *discounted value process* of a trading strategy φ is the real-valued adapted process $V(\varphi) = (V_k(\varphi))_{k=0,1,\dots,T}$ given by

$$V_k(\varphi) := \varphi_k^0 S_k^0 + \vartheta_k^{\mathrm{tr}} S_k.$$

- Note that we could also define $V_k(\varphi) = \widetilde{V}_k(\varphi)/\widetilde{S}_k^0$ with $\widetilde{V}_k(\varphi) = \varphi_k^0 \widetilde{S}_k^0 + \vartheta_k^{\mathrm{tr}} \widetilde{S}_k$. This is obvious from the definition, but we will see that a similar relationship does not hold for other related processes.
- $V_0(\varphi) = \varphi_0^0$ corresponds to the initial amount in the bank account before we start trading.
- Before we reach time $k = 1$, we need to decide on φ_1^0 and ϑ_1 and then at time $k = 1$ we have $V_k(\varphi) = \varphi_k^0 S_k^0 + \vartheta_k^{\mathrm{tr}} S_k$ etc.

Definition 3 (Discounted gains process)

The *discounted gains process* of trading strategy φ is the real-valued adapted process $G(\varphi) = (G_k(\varphi))_{k=0,1,\dots,T}$ given by

$$G_k(\varphi) := \sum_{j=1}^{k} \vartheta_j^{\mathrm{tr}} \Delta S_j.$$

- Changes in the discounted value processes due to changes in the discounted bank account are zero.
- Note that unlike before we do not have that $G_k(\varphi) = \widetilde{G}_k(\varphi)/\widetilde{S}_k^0$ with $\widetilde{G}_k(\varphi) = \sum_{j=1}^{k} \vartheta_j^{\mathrm{tr}} \Delta \widetilde{S}_j$. What is the message here? When we do any calculations, it is better to stick with either the discounted prices or the undiscounted prices. We opt for the discounted prices and the reason why will become apparent later.

Definition 4 (Discounted cost process)

The *discounted cost process* of trading strategy φ is the real-valued adapted process $C(\varphi) = (C_k(\varphi))_{k=0,1,\ldots,T}$ given by

$$C_k(\varphi) := V_k(\varphi) - G_k(\varphi).$$

- Where else can the change in $V(\varphi)$ come from if not from the changes in prices of the risky assets represented by $G(\varphi)$? Only external funding.
- We have that $V_0(\varphi) = \varphi_0^0$ and $G_0(\varphi) = 0$, thus $C_0(\varphi) = \varphi_0^0$, which is just the initial investment placed into the bank account before trading starts.

We will be dealing almost exclusively with *self-financing strategies* which do not admit any external funding except for time $k = 0$.

Definition 5 (Self-financing trading strategy)

A trading strategy φ is called *self-financing* if its cost process $C(\varphi)$ is constant over time and hence equal to $C_0(\varphi) = V_0(\varphi) = \varphi_0^0$ P-a.s., the initial investment into the bank account.

Why are we interested in such strategies? We will try to price financial instruments by recreating/replicating their price processes by investing smartly in the risky assets. If these strategies are self-financing, then the price must be equal to the initial amount put into the bank account before trading starts. Any other price would lead to the possibility of riskless profit. This will be discussed in a formal setting later.

Basic Results for Self-financing Trading Strategies

Let $\varphi = (\varphi^0, \vartheta)$ be a self-financing strategy. Then

- $C(\varphi)$ is constant by definition, therefore $\Delta C_{k+1}(\varphi) = 0$ P-a.s.
- φ is fully specified by the initial investment $V_0(\varphi)$ and the holdings in the risky assets, ϑ.
- $V(\varphi) = V_0(\varphi) + G(\varphi) = V_0(\varphi) + G(\vartheta)$.

Further Concepts from Probability Theory

Definition 6 (Independent events)

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Two events A, B (i.e. two sets from $\mathcal{F}$) are said to be *independent* if $P[A \cap B] = P[A]P[B]$.

Definition 7 (Independent σ-algebras)

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Two σ-algebras $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ are said to be *independent* if $P[A \cap B] = P[A]P[B]$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Definition 8 (Independent random variables)

Two random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$ are said to be *independent* if $\sigma(X)$ and $\sigma(Y)$ are independent.

Definition 9 (Stopping time)

A random variable $\tau : \Omega \to \{0, 1 \ldots, T\}$, $T \in \mathbb{N}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\ldots,T}$ is called an $\mathbb{F}$-*stopping time* if $\{\tau \leq j\} \in \mathcal{F}_j$ for all $j = 0, 1 \ldots T$.

- Equivalently, we have that τ is a stopping time if $\{\tau = j\} \in \mathcal{F}_j$ for all j.
- All this means is that for a random variable τ to be a stopping time, we want to be able to tell at any point in time whether this "time to stop a process" has occurred or not.
- Stopping times are typically induced by a combination of current and past values of some process(es) defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$.
- Note also that we do not require that τ is $\mathcal{F}_j$-measurable for any j.
- A typical way to show that a random variable is a stopping time is to decompose the set $\{\tau \leq j\}$ into at most countable union of sets whose measurability with respect to $\mathcal{F}_j$ is easy to show.

Example 10 (Stopping time)

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\ldots,T}$ be a filtered probability space and $X = (X_k)_{k=0,1,\ldots,T}$ for $T \in \mathbb{N}$ be an $\mathbb{F}$-adapted stochastic process. Define

$$\tau(\omega) := \inf\{k = 0, 1, \ldots, T \,:\, X_k(\omega) \in B\} \wedge T,$$

for a $B \in \mathcal{B}(\mathbb{R})$. This τ is an $\mathbb{F}$-stopping time, which we can show in two ways:

1. We show that $\{\tau = k\} \in \mathcal{F}_k$ for all k:
   $\{\tau = k\} = \left(\bigcap_{n=0}^{k-1}\{X_n \notin B\}\right) \cap \{X_k \in B\} = \left(\bigcap_{n=0}^{k-1}\{X_n \in B^c\}\right) \cap \{X_k \in B\}$
2. We show that $\{\tau \leq k\} \in \mathcal{F}_k$ for all k:
   $\{\tau \leq k\} = \bigcup_{n=0}^{k}\{X_k \in B\}$

We see that sometimes one is simpler than the other.

Further Concepts from Probability Theory

Definition 11 (Martingale)

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\ldots,T}$ be a filtered probability space. A real-valued stochastic process $X = (X_k)_{k=0,1,\ldots,T}$ is called a *martingale* (with respect to $\mathbb{F}$ and P) if

1. X is adapted to $\mathbb{F}$,
2. $X_k \in L^1(P)$ for all $k = 0, 1, \ldots, T$,
3. X satisfies the *martingale property*, i.e. $E[X_l \mid \mathcal{F}_k] = X_k$ P-a.s. for $k \leq l$.

- If the equality in 3 is exchanged by "$\leq$" ("$\geq$") one gets the definition of a *supermartingale* (*submartingale*).
- Most frequently we will be interested in the martingale property with respect to the natural filtration (representing the process' past), but this does not have to be so.
- Note that the set of all supermartingales (submartingales) on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a superset of the set of all martingales on the same space. We will encounter theorems formulated e.g. for submartingales, but these also hold for martingales.

Examples of Martingales

Example 12 (Processes that are martingales)

- A constant process X defined by $X_k = c$ for a $c \in \mathbb{R}$.
- A simple random walk X defined by $X_k = \sum_{n=0}^{k} Z_n$ with Z_k i.i.d. random variables taking the value 1 with probability 0.5 and the value -1 with probability 0.5.
- A stochastic process X defined by $X_k = \prod_{n=0}^{k} Z_n$ for non-negative i.i.d random variables Z_k with $E[Z_k] = 1$.

Example 13 (Processes that are not martingales)

- Any non-constant deterministic process, e.g. X defined by $X_k = 2k + 1$.
- A *drifted* random walk X defined by $X_k = 2k + \sum_{n=0}^{k} Z_n$ for Z_k as before.
- A random walk X defined by $X_k = \sum_{n=0}^{k} Z_n$ for i.i.d. $Z_k \notin L^1$ (for instance Z_k have Cauchy distribution).
- Any non-adapted stochastic process.

Thank you for your attention!