Coordinator
Martin Stefanik

Mathematical Foundations for Finance

Exercise sheet 2

Exercise 2.1 Let us assume the basic multiplicative model for our financial market $(\widetilde{S}^0, \widetilde{S}^1)$. We start on a probability space $(\Omega, \mathcal{F}, P)$ with random variables $r_1, \dots, r_T > -1$ and $Y_1, \dots, Y_T > 0$ for a $T \in \mathbb{N}$. Define for $k = 0, \dots, T$

$$\widetilde{S}^0_k := \prod_{j=1}^{k} (1 + r_j), \quad \widetilde{S}^1_k := S^1_0 \prod_{j=1}^{k} Y_j,$$

with a constant $S^1_0 > 0$.

(a) A natural filtration to use in this model is the filtration generated by $Y = (Y_k)_{k=1,\dots,T}$ and $r = (r_k)_{k=1,\dots,T}$, i.e. the one given by

$$\begin{aligned} \mathcal{F}'_0 &= \{\emptyset, \Omega\}, \\ \mathcal{F}'_k &= \sigma(Y_1, \dots, Y_k, r_1, \dots, r_k) \quad \text{for } k = 1, \dots, T. \end{aligned}$$

Show that if one assumes r to be predictable with respect to this filtration, then we have that $\mathcal{F}'_k = \mathcal{F}_k := \sigma(\widetilde{S}^1_0, \widetilde{S}^1_1, \dots, \widetilde{S}^1_k)$ for all $k = 0, \dots, T$.
(*Hint: if $\mathcal{A}$ and $\mathcal{B}$ are two collections of subsets of Ω, then $\sigma(\mathcal{A} \cup \mathcal{B}) = \sigma(\sigma(\mathcal{A}) \cup \sigma(\mathcal{B}))$*)

(b) Recall that we call a strategy $\varphi = (\varphi^0, \vartheta)$ self-financing if its discounted cost process $C(\varphi)$ is constant over time. Show that the notion of self-financing strategy does not depend on whether we work with the discounted price processes S^0 and S^1 or the undiscounted processes $\widetilde{S}^0$ and $\widetilde{S}^1$, i.e. show that the discounted cost process $C(\varphi)$ is constant over time if and only if the undiscounted cost process $\widetilde{C}(\varphi)$, determined by

$$\Delta\widetilde{C}_{k+1}(\varphi) := \widetilde{C}_{k+1}(\varphi) - \widetilde{C}_k(\varphi) = (\varphi^0_{k+1} - \varphi^0_k)\widetilde{S}^0_k + (\vartheta_{k+1} - \vartheta_k)\widetilde{S}^1_k,$$

is constant over time.

(c) Use the result in (b) to conclude that the notion of self-financing strategy is numeraire-invariant, i.e. that it does not matter for this definition whether the discounted price processes are defined as $S^0 := \widetilde{S}^0/\widetilde{S}^0$ and $S^1 := \widetilde{S}^1/\widetilde{S}^0$, or $\bar{S}^0 := \widetilde{S}^0/\widetilde{S}^1$ and $\bar{S}^1 := \widetilde{S}^1/\widetilde{S}^1$.

Solution 2.1

(a) As suggested in the Remark on page 7 of the lecture notes, we proceed by induction:

1. We show that $\mathcal{F}'_k = \mathcal{F}_k$ for $k = 0$:

   We clearly have that $\mathcal{F}'_0 = \mathcal{F}_0$ since $\widetilde{S}^1_0$ is a constant random variable that generates the trivial σ-algebra $\{\emptyset, \Omega\}$.

2. The induction step – $\mathcal{F}'_k = \mathcal{F}_k \implies \mathcal{F}'_{k+1} = \mathcal{F}_{k+1}$:

   We have that

$$\begin{aligned} \mathcal{F}'_{k+1} &= \sigma\big(\sigma(r_1, \dots, r_k, Y_1, \dots, Y_k) \cup \sigma(r_{k+1}) \cup \sigma(Y_{k+1})\big) \\ &= \sigma\big(\sigma(\widetilde{S}^1_1, \dots, \widetilde{S}^1_k) \cup \sigma(r_{k+1}) \cup \sigma(Y_{k+1})\big) \\ &= \sigma\big(\sigma(\widetilde{S}^1_1, \dots, \widetilde{S}^1_{k+1}) \cup \sigma(r_{k+1})\big) \end{aligned}$$

where for the second equality we have used our induction hypothesis that $\mathcal{F}'_k = \mathcal{F}_k$ and for the third we used the hint and the fact that

$$\sigma(\widetilde{S}^1_1, \dots, \widetilde{S}^1_k, Y_{k+1}) = \sigma(\widetilde{S}^1_1, \dots, \widetilde{S}^1_k, \widetilde{S}^1_{k+1})$$

since $\widetilde{S}^1_{k+1} = \widetilde{S}^1_k Y_{k+1}$. Furthermore, since r_{k+1} is $\mathcal{F}'_k$-measurable, and thus also $\mathcal{F}_k$-measurable by our induction hypothesis, we must have that $\sigma(r_{k+1}) \subseteq \mathcal{F}_k$. This gives

$$\mathcal{F}'_{k+1} = \sigma\big(\mathcal{F}_{k+1} \cup \sigma(r_{k+1})\big) = \sigma(\mathcal{F}_{k+1}) = \mathcal{F}_{k+1}$$

and concludes the proof.

(b) Using the definition of the incremental cost for a strategy φ from the lecture, $\Delta C_{k+1}(\varphi) = C_{k+1}(\varphi) - C_k(\varphi)$, we get that

$$C_{k+1}(\varphi) = \Delta C_{k+1}(\varphi) + C_k(\varphi) = C_0(\varphi) + \sum_{j=1}^{k+1} \Delta C_j(\varphi).$$

Therefrom it is clear that the cost process $C_k(\varphi)$ is constant over time (and equal to the initial investment into the bank account, φ^0_0) if and only if we have for each k that $\Delta C_{k+1}(\varphi) = 0$ P-a.s. Since we have that

$$\Delta C_k(\varphi) = (\varphi^0_{k+1} - \varphi^0_k) S^0_k + (\vartheta_{k+1} - \vartheta_k) S^1_k,$$

the self-financing condition boils down to

$$0 = (\varphi^0_{k+1} - \varphi^0_k) S^0_k + (\vartheta_{k+1} - \vartheta_k) S^1_k$$

P-a.s. for each k. Multiplying both sides of the equation by $\widetilde{S}^0_k$ we obtain the same condition for the undiscounted prices, i.e.

$$0 = (\varphi^0_{k+1} - \varphi^0_k) \widetilde{S}^0_k + (\vartheta_{k+1} - \vartheta_k) \widetilde{S}^1_k = \Delta \widetilde{C}_{k+1}(\varphi).$$

Since we also have that

$$\widetilde{C}_{k+1}(\varphi) = \widetilde{C}_0(\varphi) + \sum_{j=1}^{k+1} \Delta \widetilde{C}_j(\varphi),$$

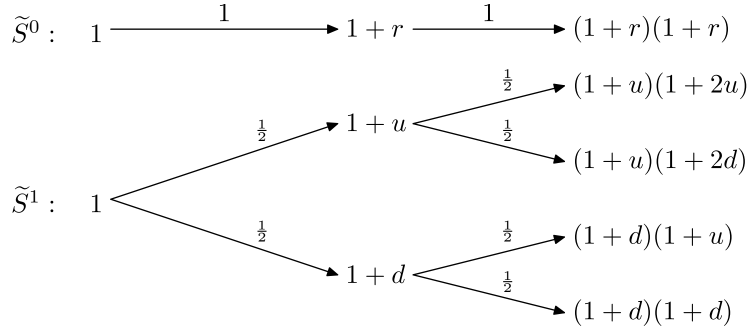
this indeed shows that the discounted cost process is P-a.s. constant if and only if the undiscounted process is.

(c) We have shown in (b) that a strategy φ is self-financing if and only if we have that

$$(\varphi^0_{k+1} - \varphi^0_k) \widetilde{S}^0_k + (\vartheta_{k+1} - \vartheta_k) \widetilde{S}^1_k = 0.$$

P-a.s. for all k. In order to express this condition in either version of the discounted processes S^0 and S^1, it is enough to divide both sides of the equation by $\widetilde{S}^1_k$ or $\widetilde{S}^0_k$. Since $\widetilde{S}^1_k, \widetilde{S}^0_k > 0$, the self-financing condition still holds after this operation.

Exercise 2.2 Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ given by the following trees, where the numbers beside the branches denote transition probabilities.



Intuitively, this means that the volatility of $\widetilde{S}^1$ increases if the stock price increases in the first period. Assume that u, $r \geq 0$ and $-0.5 < d \leq 0$.

(a) Construct for this setup a multiplicative model consisting of a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$, two random variables Y_1 and Y_2 and two adapted stochastic processes $\widetilde{S}^0$ and $\widetilde{S}^1$ such that $\widetilde{S}^1_k = \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$.

(b) For which values of u and d are Y_1 and Y_2 *uncorrelated*?

(c) For which values of u and d are Y_1 and Y_2 *independent*?

(d) For which values of u, r and d is the discounted stock process S^1 a P-martingale?

Solution 2.2

(a) We construct the canonical model for this setup, a path space. Let $\Omega := \{-1, 1\}^2$, take $\mathcal{F} := 2^\Omega$ and define P by

$$P\left[\{(x_1, x_2)\}\right] := p_{x_1} p_{x_1, x_2}\,,$$

where $p_1 = p_{-1} := 1/2$ and $p_{1,1} = p_{1,-1} = p_{-1,1} = p_{-1,-1} := 1/2$. Next, define Y_1 and Y_2 by $Y_1((1,1)) = Y_1((1,-1)) := 1 + u$, $Y_1((-1,1)) = Y_1((-1,-1)) := 1 + d$ and $Y_2((1,1)) := 1 + 2u$, $Y_2((1,-1)) := 1 + 2d$, $Y_2((-1,1)) := 1 + u$, $Y_2((-1,-1)) := 1 + d$. Finally, define $\widetilde{S}^0$ and $\widetilde{S}^1$ by $\widetilde{S}^0_k := (1+r)^k$ and $\widetilde{S}^1_k := \prod_{j=1}^k Y_j$ for $k = 0, 1, 2$ and set $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_1 := \sigma(Y_1) = \{\emptyset, \{(1,1), (1,-1)\}, \{(-1,1), (-1,-1)\}, \Omega\}$ and $\mathcal{F}_2 := \sigma(Y_1, Y_2) = 2^\Omega = \mathcal{F}$.

(b) Y_1 and Y_2 are uncorrelated if and only if $E\left[Y_1 Y_2\right] = E\left[Y_1\right] E\left[Y_2\right]$. Set $c := (u + d)/2$ to simplify the notation. Then we have

$$E\left[Y_1\right] = 1 + c \quad \text{and} \quad E\left[Y_2\right] = 1 + \frac{3}{2}c\,,$$
$$E\left[Y_1 Y_2\right] = \frac{1+u}{2}(1 + 2c) + \frac{1+d}{2}(1 + c) = (1 + c)^2 + \frac{1+u}{2}c\,.$$

Hence, we have

$$E\left[Y_1 Y_2\right] - E\left[Y_1\right] E\left[Y_2\right] = (1 + c)^2 + \frac{1+u}{2}c - \left((1 + c)^2 + (1 + c)\frac{c}{2}\right)$$
$$= (u - c)\frac{c}{2}\,.$$

Since $d \leq 0 \leq u$, we have

$$(u - c)\frac{c}{2} = 0 \quad \Leftrightarrow \quad c = 0 \ \text{ or } \ u - c = 0 \quad \Leftrightarrow \quad d = -u\,.$$

In conclusion, Y_1 and Y_2 are uncorrelated if and only if $d = -u$.

(c) Since independence of two random variables implies that they are uncorrelated, we only have to consider the case in which $u = -d$. If $u = d = 0$, Y_1 and Y_2 are both constant random variables and hence independent. Otherwise, if $u > 0$ we have on the one hand

$$P\left[Y_1 = 1 + u, Y_2 = 1 + u\right] = 0$$

and on the other hand

$$P\left[Y_1 = 1 + u\right] P\left[Y_2 = 1 + u\right] = 1/2 \cdot 1/4 = 1/8 \neq 0\,,$$

showing that in this case Y_1 and Y_2 are not independent. In conclusion, Y_1 and Y_2 are independent if and only if $u = d = 0$.

Note: If $d = -u$ and $u \neq 0$, then Y_1 and Y_2 are uncorrelated but **not** independent.

(d) S^1 is a P-martingale if and only if

$$E\left[S_1^1 \,\middle|\, \mathcal{F}_0\right] = S_0^1 \quad P\text{-a.s.} \quad \text{and} \quad E\left[S_2^1 \,\middle|\, \mathcal{F}_1\right] = S_1^1 \quad P\text{-a.s.} \tag{1}$$

If $u = d = 0$, it is straightforward to check that S^1 is a P-martingale if and only if $r = 0$. Next, assume that $u > d$. Since $\mathcal{F}_0$ is trivial, $\mathcal{F}_1 = \sigma(Y_1)$ and $Y_1 > 0$, (1) is equivalent to

$$E\left[Y_1\right] = 1 + r \quad \text{and} \quad E\left[Y_2 \,|\, Y_1\right] = 1 + r \quad P\text{-a.s.}$$

Since Y_1 only takes two values, this is equivalent to

$$E\left[Y_1\right] = 1 + r \quad \text{and} \quad E\left[Y_2 \,|\, Y_1 = 1 + u\right] = 1 + r \quad \text{and} \quad E\left[Y_2 \,|\, Y_1 = 1 + d\right] = 1 + r\,.$$

This is equivalent to the linear system

$$\begin{aligned} 1 + (u + d)/2 &= 1 + r\,, \\ 1 + u + d &= 1 + r\,, \\ 1 + (u + d)/2 &= 1 + r\,. \end{aligned}$$

Subtracting the first from the second equation yields $(u + d)/2 = 0$, which in turn implies $r = 0$. In conclusion, S^1 is a P-martingale if and only if $r = 0$ and $d = -u$.

Exercise 2.3 Consider for a finite time horizon $T \geq 2$ a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ consisting of a bank account and one stock defined on a probability space $(\Omega, \mathcal{F}, P)$. Assume that $\widetilde{S}_0^1 = 1$ and $\widetilde{S}_k^1 > 0$ P-a.s. for all $k = 0, \ldots, T$. Fix thresholds $0 < \ell < 1 < u$ and define

$$\begin{aligned} \sigma(\omega) &:= \inf\{k = 0, \ldots, T : \, S_k^1(\omega) \leq \ell\} \wedge T, \\ \tau(\omega) &:= \inf\{k = \sigma(\omega), \ldots, T : \, S_k^1(\omega) \geq u\} \wedge T, \end{aligned}$$

where $\inf \emptyset = +\infty$ as usual. Moreover, for $k = 0, \ldots, T$ define

$$\vartheta_k(\omega) := \mathbb{1}_{\{\sigma(\omega) < k \leq \tau(\omega)\}}\,.$$

Finally define the filtration $\mathbb{F} = (\mathcal{F}_k)_{0 \leq k \leq T}$ by $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F}_k = \sigma(\widetilde{S}_i^1, \ i \leq k)$.

(a) Show that σ and τ are *stopping times*, i.e. that for all $k = 0, \ldots, T$, we have

$$\{\sigma \leq k\}, \{\tau \leq k\} \in \mathcal{F}_k\,.$$

(b) Show that ϑ is a predictable process with $\vartheta_0 = \vartheta_1 = 0$.

(c) Construct φ^0 such that $\varphi = (\varphi^0, \vartheta)$ is a self-financing strategy with $V_0(\varphi) = 0$ and derive a formula for the (discounted) value process $V(\varphi)$ involving only the discounted stock price S^1 and the stopping times σ and τ.

(d) Describe the trading strategy φ in words.

Solution 2.3

(a) We have $\{\sigma \leq 0\} = \emptyset \in \mathcal{F}_0$ and $\{\sigma \leq T\} = \Omega \in \mathcal{F}_T$. For $k = 1, \ldots, T - 1$ we have

$$\{\sigma \leq k\} = \bigcup_{j=1}^{k} \{S_j^1 \leq \ell\} \in \mathcal{F}_k\,,$$

since for $1 \leq j \leq k$ we have $\{S^1_j \leq \ell\} \in \mathcal{F}_j \subset \mathcal{F}_k$ (because S^1 is adapted) and a finite union of sets in $\mathcal{F}_k$ is in $\mathcal{F}_k$ by the definition of σ-algebra.

We have $\{\tau \leq 0\} = \emptyset \in \mathcal{F}_0$, $\{\tau \leq 1\} = \emptyset \in \mathcal{F}_1$ and $\{\tau \leq T\} = \Omega \in \mathcal{F}_T$. For $k = 2, \dots, T-1$ we have

$$\{\tau \leq k\} = \bigcup_{1 \leq i < j \leq k} \{S^1_i \leq \ell, S^1_j \geq u\} \in \mathcal{F}_k \,,$$

since for $1 \leq i < j \leq k$ we have

$$\{S^1_i \leq \ell, S^1_j \geq u\} = \{S^1_i \leq \ell\} \cap \{S^1_j \geq u\},\ \{S^1_i \leq \ell\} \in \mathcal{F}_i \subset \mathcal{F}_k$$

and $\{S^1_j \geq u\} \in \mathcal{F}_j \subset \mathcal{F}_k$ (because S^1 is adapted) and a finite union of sets in $\mathcal{F}_k$ is in $\mathcal{F}_k$ by the definition of σ-algebra.

(b) Since $\sigma \geq 1$ P-a.s., it follows immediately from the definition of ϑ that $\vartheta_0 = \vartheta_1 = 0$. It remains to show that for all $k = 2, \dots, T$ we have $\{\vartheta_k = 1\} \in \mathcal{F}_{k-1}$. Indeed, for $k = 2, \dots, T$ we have

$$\{\vartheta_k = 1\} = \{\sigma \leq k-1, \tau \geq k\} = \{\sigma \leq k-1\} \setminus \{\tau \leq k-1\} \in \mathcal{F}_{k-1},$$

since σ and τ are stopping times.

(c) First, we calculate the (discounted) gains process associated with ϑ, which is for $k = 0, \dots, T$ given by

$$G_k(\vartheta) = \sum_{j=1}^{k} \mathbb{1}_{\{\sigma < j \leq \tau\}} \Delta S^1_j = S^1_{\tau \wedge k} - S^1_{\sigma \wedge k} \quad P\text{-a.s.}$$

By formula (2.8) in the lecture notes, the trading strategy $\varphi = (\varphi^0, \vartheta)$ is self-financing if and only if for $k = 0, \dots, T$ we have

$$V_k(\varphi) = V_0(\varphi) + G_k(\vartheta) \quad P\text{-a.s.}$$

Using that $V_k(\varphi) = \varphi^0_k + \vartheta_k S^1_k$ by definition and $V_0(\varphi) = 0$ by assumption, we get $\varphi^0_0 = 0$ and for $k = 1, \dots, T$ we have P-a.s.

$$\begin{aligned} \varphi^0_k &= G_k(\vartheta) - \vartheta_k S^1_k = S^1_{\tau \wedge k} - S^1_{\sigma \wedge k} - \mathbb{1}_{\{\sigma < k \leq \tau\}} S^1_k \\ &= -S^1_\sigma \mathbb{1}_{\{\sigma < k \leq \tau\}} + (S^1_\tau - S^1_\sigma) \mathbb{1}_{\{k > \tau\}} \,. \end{aligned}$$

Moreover, we clearly have for $k = 0, \dots, T$

$$V_k(\varphi) = G_k(\vartheta) = S^1_{\tau \wedge k} - S^1_{\sigma \wedge k} \quad P\text{-a.s.}$$

(d) This strategy can be described as a "buy low and sell high" strategy. When the discounted price of the stock falls below ℓ, one borrows money to buy a share of the stock. As soon as the discounted price of the stock is above u, one sells the share, pays back the loan and stores the difference between buying and selling price in the bank account.