ETH zürich

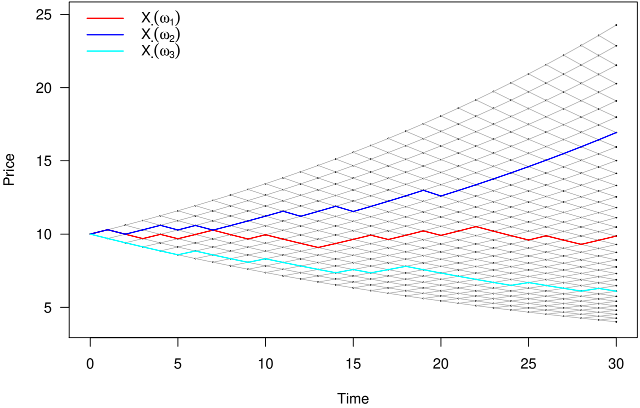
Mathematical Foundations for Finance

Exercise 3

Martin Stefanik
ETH Zurich

Illustration of the Binomial model

Evolution of the stock price in a (multiplicative) binomial model with parameters $\widetilde{S}_0^1 = 10$, $u = -d = 0.03$.



Definition 1 (Martingale)

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ be a filtered probability space.
A real-valued stochastic process $X = (X_k)_{k=0,1,\dots,T}$ is called a *martingale* (with respect to $\mathbb{F}$ and P) if

1. X is adapted to $\mathbb{F}$,
2. $X_k \in L^1(P)$ for all $k = 0, 1, \dots, T$,
3. X satisfies the *martingale property*, i.e. $E[X_l \mid \mathcal{F}_k] = X_k$ P-a.s. for $k \leq l$.

If the equality in 3 is substituted by "$\leq$" ("$\geq$") one gets the definition of a *supermartingale* (*submartingale*).

Local Martingales in Discrete Time

Definition 2 (Local martingale)

An adapted stochastic process $X = (X_k)_{k=0,1,\dots,T}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $X_0 = 0$ is called a *local martingale* (with respect to P and $\mathbb{F}$) if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to T such that for each $n \in \mathbb{N}$ the stopped process $X^{\tau_n} = (X_{k\wedge\tau_n})_{k=0,1,\dots,T}$ is a $(P, \mathbb{F})$-martingale.

- We also call $(\tau_n)_{n\in\mathbb{N}}$ a *localizing sequence.*
- Notice that even though we are in finite (discrete) time, the localizing sequence is countably infinite.
- All martingales are local martingales – take deterministic $\tau_n = n \wedge T$.
- We typically refer to local martingales that are not martingales as *strict local martingales.*
- Strict local martingales in discrete time typically fail to be integrable (and therefore cannot be martingales), but in continuous time there exist integrable strict local martingales.

Discrete Time Stochastic Integral

Definition 3 (Stochastic integral)

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ be a filtered probability space. Let $X = (X_k)_{k=0,1,\dots,T}$ be an $\mathbb{F}$-adapted and $\vartheta = (\vartheta_k)_{k=0,1,\dots,T}$ an $\mathbb{F}$-predictable process. Then we define the *stochastic integral process* of ϑ with respect to X, $\vartheta \bullet X = (\vartheta \bullet X_k)_{k=0,1,\dots,T}$ by

$$\vartheta \bullet X_k = \sum_{j=1}^{k} \vartheta_j \Delta X_j \quad \text{for } k = 0, 1, \dots, T.$$

- Even though the process is well-defined for any such process X in discrete time, we need impose some regularity on the integrators in continuous time in order to define it there.
- We will use stochastic integral exclusively with respect to local martingales, submartingales and supermartingales.

Some Results for Discrete Time Stochastic Integral

Note that even though some of these results hold in continuous time as well, try to associate them with discrete time only. We will provide a similar "list" for continuous time as well.

1. If X is a local martingale then $\vartheta \bullet X$ is a local martingale.
2. If X is a martingale and ϑ is bounded, then $\vartheta \bullet X$ is a martingale.
3. If X is a submartingale and $\vartheta \geq 0$ is bounded, then $\vartheta \bullet X$ is a submartingale.
4. If X is a supermartingale and $\vartheta \geq 0$ is bounded, then $\vartheta \bullet X$ is a supermartingale.
5. If X is a martingale and Ω is finite, then $\vartheta \bullet X$ is a martingale.
6. If $\vartheta \bullet X$ is uniformly bounded from below, then $\vartheta \bullet X$ is a martingale.

Equivalent Measures

Definition 4 (Absolutely continuous probability measure)

Let Q and P be two probability measures on a measurable space $(\Omega, \mathcal{F})$. Then Q is said to be *absolutely continuous* with respect to P on $\mathcal{F}$ and is denoted $Q \ll P$, if $P[A] = 0 \implies Q[A] = 0$ for all $A \in \mathcal{F}$.

Definition 5 (Equivalent probability measure)

Two probability measures Q and P on a measurable space $(\Omega, \mathcal{F})$ are said to be equivalent on $\mathcal{F}$ and is denoted $Q \approx P$, if $P[A] = 0 \iff Q[A] = 0$ for all $A \in \mathcal{F}$.

Of great importance in mathematical finance are the so-called equivalent martingale measures, which are the subject of the next definition.

Equivalent Martingale Measures

Definition 6 (Equivalent martingale measure)

Let P and Q be two probability measures on a measurable space $(\Omega, \mathcal{F})$ and let $X = (X_k)_{k=0,1,\dots,T}$ be a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. A probability measure Q equivalent to P on $\mathcal{F}$ is called an *equivalent martingale measure (EMM)* for X if X is a $(Q, \mathbb{F})$-martingale.

- How are EMMs useful? If the discounted price process of a financial derivative product, $C = (C_k)_{k=0,1,\dots,T}$ is a $(Q, \mathbb{F})$-martingale, then $C_k = E_Q[C_T \mid \mathcal{F}_k]$ for all $k = 0, 1, \dots, T$ – we can compute the price at any point in time simply by taking expectation under Q.
- This is useful, but why does it make sense to compute the prices like that? As we will see later in more detail, the prices computed this way rule out *arbitrage* under certain conditions.

Arbitrage Opportunity

Definition 7 (Arbitrage opportunity)

An *arbitrage opportunity* is an admissible self-financing strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with zero initial wealth, with $V_T(\varphi) \geq 0$ P-a.s. and with $P[V_T(\varphi) > 0] > 0$. The financial market $(\Omega, \mathcal{F}, \mathbb{F}, P, S^0, S^1)$ or shortly S is called *arbitrage-free* if there exist no arbitrage opportunities. Sometimes one also says that S satisfies (NA).

- Admissible so that we exclude strategies that we would not be able to carry out anyway (such as the doubling strategy).
- Self-financing and with zero initial investment at time $k = 0$ so that no external financing is needed.
- $V_T(\varphi) \geq 0$ P-a.s. so that we do not lose money P-a.s.
- $P[V_T(\varphi) > 0] > 0$ so that we stand a chance of making a gain.

Thank you for your attention!