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Mathematical Foundations for Finance

Exercise sheet 3

Exercise 3.1 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ and $X = (X_k)_{k=0,1,\dots,T}$ a martingale with respect to $\mathbb{F}$ and P.

(a) Show that for a bounded, convex function $f : \mathbb{R} \to \mathbb{R}$ the process $f(X) = (f(X_k))_{k=0,1,\dots,T}$ is a submartingale with respect to $\mathbb{F}$ and P. What can you say if f is not bounded?
Hint: Any convex function on $\mathbb{R}^n$ is continuous on the interior of the set where it is finite.

(b) Let $\vartheta = (\vartheta_k)_{k=0,1,\dots,T}$ with $\vartheta_0 = 0$ be a bounded, nonnegative, $\mathbb{F}$-predictable process. Show that the stochastic integral process $\vartheta \bullet f(X)$ defined by

$$\vartheta \bullet f(X)_k = \sum_{j=1}^{k} \vartheta_j \Delta f(X_j)$$

is a submartingale with respect to $\mathbb{F}$ and P.
Hint: We have proved in the lecture that the stochastic integral process of a similar predictable process with respect to a martingale is martingale.

(c) Let τ be a stopping time with respect to $\mathbb{F}$. Show that the stopped process $f(X)^\tau = (f(X)^\tau_k)_{k=0,1,\dots,T}$ defined by $f(X)^\tau_k = f(X_{k\wedge\tau})$ is a submartingale with respect to $\mathbb{F}$.
Hint: Try to express the stopped process as an appropriate stochastic integral process.

(d) Let $Y = (Y_k)_{k=0,1,\dots,T}$ be an adapted, integrable process. Show that Y is a martingale if and only if $E\,[Y_{k+1}\,|\,\mathcal{F}_k] = Y_k$ P-a.s. for $k = 0, 1, \dots, T-1$.

Solution 3.1

(a) Since f is bounded, we have that $E[|f(X_k)|] < \infty$ for $k = 0, 1, \dots, T$. This gives the integrability condition. Adaptedness of $f(X)$ is clear since f is continuous and X is an adapted process by definition. Applying Jensen's inequality for conditional expectation and then the martingale property of X, we get that

$$E[f(X_l)\,|\,\mathcal{F}_k] \ge f\big(E[X_l\,|\,\mathcal{F}_k]\big) = f(X_k)$$

P-a.s. for $k \le l$, which concludes the proof.

If f is not bounded, then the assumption that $f(X)$ is integrable (meaning that $E[|f(X_k)|] < \infty$ for all $k = 0, 1, \dots, T$) is still sufficient to obtain the same conclusion.

(b) Since ϑ is a bounded, predictable process and $f(X)$ is integrable, $\vartheta \bullet f(X)$ is integrable and adapted. To show the submartingale property, we compute

$$\begin{aligned} E[\vartheta \bullet f(X)_{k+1} - \vartheta \bullet f(X)_k\,|\,\mathcal{F}_k] &= E\big[\vartheta_{k+1}\big(f(X_{k+1}) - f(X_k)\big)\,|\,\mathcal{F}_k\big] \\ &= \vartheta_{k+1} E\big[f(X_{k+1}) - f(X_k)\,|\,\mathcal{F}_k\big] \overset{P\text{-a.s.}}{\ge} 0. \end{aligned}$$

The first equality is clear from the definition of the stochastic integral process, the second uses that ϑ_{k+1} is $\mathcal{F}_k$-measurable and bounded and the last inequality follows from the fact that $\vartheta \ge 0$ P-a.s. and the submartingale property of $f(X)$.

(c) Choosing $\vartheta_k = \mathbb{1}_{\{\tau \geq k\}}$, we can simply write

$$f(X)^\tau_k = f(X_0) + \sum_{j=1}^{k} \vartheta_j \Delta f(X_j) = f(X_0) + \vartheta_k \bullet f(X_k).$$

Since the former choice of ϑ precisely puts us into the setup of (b), this is enough to show that the stopped process $f(X)^\tau$ is indeed a submartingale.

(d) We first show the easier implication, ($\implies$). Assuming that Y is a martingale with respect to $\mathbb{F}$ and P, $E[Y_{k+1} \,|\, \mathcal{F}_k] = Y_k$ holds P-a.s. for $k = 0, 1, \dots, T-1$ as a special case of the martingale property of Y (i.e. $E[Y_l \,|\, \mathcal{F}_k] = Y_k$ P-a.s. for all $k \leq l$) for $l = k+1$.

For ($\impliedby$) we note that Y is adapted and integrable by assumption and we therefore only need to show that the martingale property of Y follows from $E[Y_{k+1} \,|\, \mathcal{F}_k] = Y_k$ P-a.s. for $k = 0, 1, \dots, T-1$. The martingale property of Y clearly holds for $k = l$ since Y is adapted by assumption. For any $l \in \{0, 1, \dots, T\}$ and $k < l$ we have that

$$\begin{aligned} E[Y_l \,|\, \mathcal{F}_k] &= E[E[Y_l \,|\, \mathcal{F}_{l-1}] \,|\, \mathcal{F}_k] = E[Y_{l-1} \,|\, \mathcal{F}_k] = E[E[Y_{l-1} \,|\, \mathcal{F}_{l-2}] \,|\, \mathcal{F}_k] \\ &= E[Y_{l-2} \,|\, \mathcal{F}_k] = \dots = E[Y_{k+1} \,|\, \mathcal{F}_k] = Y_k, \end{aligned}$$

where for the odd equalities we have used the tower property of conditional expectation and for the even ones the assumption that $E[Y_{k+1} \,|\, \mathcal{F}_k] = Y_k$ P-a.s. for $k = 0, 1, \dots, T-1$.

Exercise 3.2 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a market modeled by a *binomial model*. More precisely, let the undiscounted price processes of the assets in our market be defined by

$$\begin{aligned} \widetilde{S}^0_k &= (1+r)^k \qquad \text{for } k = 0, 1, \dots, T, \\ \frac{\widetilde{S}^1_{k+1}}{\widetilde{S}^1_k} &= Y_{k+1} \qquad \text{for } k = 0, 1, \dots, T-1, \end{aligned}$$

where the Y_k are i.i.d. random variables taking values $1+u$ with probability $p \in (0,1)$ and $1+d$ with probability $1-p$. Assume furthermore that $u > d > -1$ and $r > -1$.

(a) Suppose that $r \leq d$. Show that in this case the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits *arbitrage* by explicitly constructing an *arbitrage opportunity.*

(b) Suppose that $r \geq u$. Show that also in this case the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits *arbitrage* by explicitly constructing an *arbitrage opportunity.*

Solution 3.2

(a) If $r \leq d$, the stock grows in each state of the world and in all trading periods at least as fast as the bank account, but in some states of the world faster since $u > d$. In mathematical terms, this means that we have for $k = 1, \dots, T$ that

$$Y_k \geq 1 + r \quad P\text{-a.s.} \quad \text{and} \quad P[Y_k > 1 + r] > 0\,.$$

In terms of the discounted stock price S^1, this means that we have for $k = 1, \dots, T$ that

$$S^1_k \geq S^1_{k-1} \quad P\text{-a.s.} \quad \text{and} \quad P\left[S^1_k > S^1_{k-1}\right] > 0\,. \tag{1}$$

Therefore, an obvious arbitrage opportunity consists in borrowing money at time 0 from the bank account to buy, say, one stock and holding the stock until the time horizon T. In mathematical terms, this means that we consider the strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$, where ϑ is given by

$\vartheta_0 = 0$ and $\vartheta_k = 1$, $k = 1, \dots, T$, which is deterministic and therefore predictable. Moreover, by formula (1.2.8) in the lecture notes, we have for $k = 1, \dots, T$ that

$$V_k(\varphi) = G_k(\vartheta) = \sum_{j=1}^{k} (1 \times \Delta S_j) = S_k^1 - S_0^1 \quad P\text{-a.s.}$$

Hence, by (1) we may deduce on the one hand that $V(\varphi) \geq 0$, whence φ is 0-admissible and thus admissible, and on the other hand that $P\left[V_T(\varphi) > 0\right] > 0$, whence φ is an arbitrage opportunity.

(b) If $r \geq u$, the stock grows in each state of the world and in all trading periods at most as fast as the bank account, but in some states of the world more slowly since $u > d$. In mathematical terms, this means that for $k = 1, \dots, T$, we have

$$Y_k \leq 1 + r \quad P\text{-a.s.} \quad \text{and} \quad P\left[Y_k < 1 + r\right] > 0\,.$$

In terms of the discounted stock price S^1, this means that for $k = 1, \dots, T$, we have

$$S_k^1 \leq S_{k-1}^1 \quad P\text{-a.s.} \quad \text{and} \quad P\left[S_k^1 < S_{k-1}^1\right] > 0. \tag{2}$$

Therefore, the obvious arbitrage opportunity consists in short-selling, say, one stock at time 0 and investing the money into the bank account until the time horizon T. In mathematical terms, this means that we consider the strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$, where ϑ is given by $\vartheta_0 = 0$ and $\vartheta_k = -1$, $k = 1, \dots, T$, which is deterministic and therefore a fortiori predictable. Moreover, by formula (1.2.8) in the lecture notes, for $k = 1, \dots, T$, we have

$$V_k(\varphi) = G_k(\vartheta) = \sum_{j=1}^{k} (-1 \times \Delta S_j) = S_0^1 - S_k^1 \quad P\text{-a.s.}$$

Hence, by (2) we may deduce on the one hand that $V(\varphi) \geq 0$, whence φ is 0-admissible and a fortiori admissible, and on the other hand that $P\left[V_T(\varphi) > 0\right] > 0$, whence φ is an arbitrage opportunity.

Exercise 3.3 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a *trinomial model*. This is like a binomial model a special case of a *multinomial model*, and the distribution of Y_k under P is given by

$$Y_k = \begin{cases} 1 + d & \text{with probability } p_1 \\ 1 + m & \text{with probability } p_2 \\ 1 + u & \text{with probability } p_3 \end{cases}$$

where p_1, p_2, $p_3 > 0$, $p_1 + p_2 + p_3 = 1$ and $-1 < d < m < u$. Here d, m and u are mnemonics for *down*, *middle* and *up*.

(a) Assume that $d = -0.5$, $m = 0$, $u = 0.25$ and $r = 0$. For $T = 1$, consider an arbitrary self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \theta)$. Show that if the total gain $G_1(\theta)$ at time $T = 1$ is nonnegative P-a.s., then

$$P[G_1(\theta) = 0] = 1.$$

What does this property imply?

(b) Show that S^1 is arbitrage-free by constructing an *equivalent martingale measure* (EMM) for S^1.
Hint: A probability measure Q equivalent to P on $\mathcal{F}_1$ can be uniquely described by a probability vector $(q_1, q_2, q_3) \in (0,1)^3$, where $q_k = Q\left[Y_1 = 1 + y_k\right]$, $k = 1, 2, 3$, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$. (A probability vector in $\mathbb{R}^n$, $n \in \mathbb{N}$ is a nonnegative vector in $\mathbb{R}^n$ whose coordinates sum up to 1.)

(c) Assume now that $d = -0.01$, $m = 0.01$, $u = 0.03$ and $r = 0.01$. For $T = 2$, give a parametrization of all EMMs for S^1.
Hint: A probability measure Q equivalent to P on $\mathcal{F}_2$ can be uniquely described by four probability vectors (q_1, q_2, q_3), $(q_{j,1}, q_{j,2}, q_{j,3}) \in (0,1)^3$, $j = 1, 2, 3$, where $q_j = Q[Y_1 = 1 + y_j]$ and $q_{j,k} = Q[Y_2 = 1 + y_k \mid Y_1 = 1 + y_j]$, $j, k = 1, 2, 3$, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$.

Solution 3.3

(a) Let us compute the total gain $G_1(\theta)$ at time $T = 1$:

$$G_1(\theta) = \theta_1^1 \Delta S_1^1 = \theta_1^1 (S_1^1 - S_0^1) = \theta_1^1 S_0^1 \left(\frac{Y_1}{1+r} - 1 \right) = \theta_1^1 S_0^1 \times \begin{cases} \frac{u-r}{1+r} & \text{with probability } p_3, \\ \frac{m-r}{1+r} & \text{with probability } p_2, \\ \frac{d-r}{1+r} & \text{with probability } p_1. \end{cases}$$

Recall that $u - r = 0.25 > 0$ and $d - r = -0.5 < 0$. Hence $P[G_1(\theta) \geq 0] = 1$ if and only if $\theta_1^1 S_0^1 = 0$. As a result, we can conclude that

$$P[G_1(\theta) \geq 0] = 1 \quad \Leftrightarrow \quad \theta_1^1 = 0 \quad \Leftrightarrow \quad P[G_1(\theta) = 0] = 1.$$

Assume now that $V_0 = 0$ and note that in this case $V_1(\varphi) = G_1(\theta)$. The above argument proves that if $V_1(\varphi) \geq 0$ P-a.s., then $V_1(\varphi) = 0$ P-a.s., and by Proposition 1.1 part 3) in the lecture notes, we know that this is equivalent to saying that S^1 is arbitrage-free.

(b) Let $(q_1, q_2, q_3) \in (0,1)^3$ be a probability vector and Q be defined by

$$Q[Y_1 = 1 + y_k] := q_k, \quad k = 1, 2, 3,$$

where $y_1 := d$, $y_2 := m$ and $y_3 := u$. Then S^1 is a Q-martingale if and only if S^1 is adapted to the considered filtration (take the filtration generated by S^1 itself), integrable (the probability space is finite here, so all random variables are integrable), and

$$\begin{aligned} E_Q\left[S_1^1\right] = S_0^1 \quad &\Leftrightarrow \quad E_Q\left[S_0^1 Y_1/(1+r)\right] = S_0^1 \quad \Leftrightarrow \quad E_Q[Y_1] = 1 + r \\ &\Leftrightarrow \quad q_1 \times (1+d) + q_2 \times (1+m) + q_3 \times (1+u) = 1 + r \\ &\Leftrightarrow \quad q_1 \times d + q_2 \times m + q_3 \times u = r \\ &\Leftrightarrow \quad -0.5 q_1 + 0 q_2 + 0.25 q_3 = 0 \\ &\Leftrightarrow \quad q_3 = 2 q_1. \end{aligned}$$

Recall that in order to make Q a probability measure, we need to have $q_1 + q_2 + q_3 = 1$; hence choosing $q_1 = 0.25$, we obtain that $q_3 = 0.5$ and $q_2 = 0.25$. Noting that $q_1, q_2, q_3 \in (0,1)$, we can also observe that Q is a probability measure equivalent to P and thus an EMM for S^1.

(c) Let $(q_1, q_2, q_3), (q_{j,1}, q_{j,2}, q_{j,3}) \in (0,1)^3$, $j = 1, 2, 3$ be probability vectors and define $Q \approx P$ on $\mathcal{F}_2 = \sigma(Y_1, Y_2)$ by

$$Q[Y_2 = 1 + y_k, Y_1 = 1 + y_j] := q_j q_{j,k}, \quad j, k = 1, 2, 3,$$

where $y_1 := d$, $y_2 := m$ and $y_3 := u$. Then S^1 is a Q-martingale if and only if it is adapted, integrable and

$$\begin{aligned} & E_Q\left[S_1^1\right] = S_0^1 \quad \text{and} \quad E_Q\left[S_2^1 \,\middle|\, \mathcal{F}_1\right] = S_1^1 \quad Q\text{-a.s.} \\ \Leftrightarrow \quad & E_Q\left[S_0^1 Y_1/(1+r)\right] = S_0^1 \quad \text{and} \quad E_Q\left[S_0^1 Y_1 Y_2/(1+r)^2 \,\middle|\, \mathcal{F}_1\right] = S_0^1 Y_1/(1+r) \quad Q\text{-a.s.} \\ \Leftrightarrow \quad & E_Q[Y_1] = 1 + r \quad \text{and} \quad E_Q[Y_2 \mid \mathcal{F}_1] = 1 + r \quad Q\text{-a.s.} \end{aligned}$$

Since $\mathcal{F}_1 = \sigma(Y_1)$ and Y_1 takes three values, the latter is equivalent to

$$E_Q[Y_1] = 1 + r \quad \text{and} \quad E_Q[Y_2 \,|\, Y_1 = 1 + y_j] = 1 + r, \quad j = 1, 2, 3,$$

which follows from the property of conditional expectation that you have proved in Exercise 1.3 (e). For the first equation we can compute

$$\begin{aligned} E_Q[Y_1] = 1 + r \quad &\Leftrightarrow \quad q_1 \times (1 + d) + q_2 \times (1 + m) + q_3 \times (1 + u) = 1 + r \\ &\Leftrightarrow \quad q_1 \times d + q_2 \times m + q_3 \times u = r \\ &\Leftrightarrow \quad -0.01q_1 + 0.01q_2 + 0.03q_3 = 0.01 \\ &\Leftrightarrow \quad -q_1 + q_2 + 3q_3 = 1\,. \end{aligned}$$

Since Q is a probability measure equivalent to P, the triplet (q_1, q_2, q_3) has to satisfy

$$\begin{cases} -q_1 + q_2 + 3q_3 & = & 1 \\ q_1 + q_2 + q_3 & = & 1 \\ q_1, q_2, q_3 \in (0, 1) \end{cases}$$

Subtracting the second equation from the first yields

$$2q_3 - 2q_1 = 0 \quad \Leftrightarrow \quad q_1 = q_3\,.$$

This in turn implies $q_2 = 1 - 2q_1$, and by the positivity constraint $0 < q_1 < 0.5$. In conclusion, $(q_1, q_2, q_3) \in (0, 1)^3$ satisfies all the required conditions if and only if it is of the form $(\lambda, 1 - 2\lambda, \lambda)$, where $0 < \lambda < 0.5$.

For the second equation note that we have again

$$E_Q[Y_2 \,|\, Y_1 = 1 + y_j] = 1 + r \quad \Leftrightarrow \quad q_{j1} \times (1 + d) + q_{j2} \times (1 + m) + q_{j3} \times (1 + u) = 1 + r.$$

Using the first part, we may thus conclude that (q_1, q_2, q_3), $(q_{j,1}, q_{j,2}, q_{j,3})$, $j = 1, 2, 3$, describe a EMM for S^1 if and only if they are of the form $(\lambda, 1 - 2\lambda, \lambda)$, $(\mu_j, 1 - 2\mu_j, \mu_j)$, where $0 < \lambda, \mu_j < 0.5$, $j = 1, 2, 3$.

Note that while the condition for the martingale property is the same in each node, this condition is satisfied by many triplets, and we are allowed to choose a different triplet for each node. In other words, transition probabilities for Q need not be homogeneous across nodes, or equivalently put, we may choose λ, μ_1, μ_2, μ_3 all different.