

Mathematical Foundations for Finance

Exercise sheet 4

Please hand in your solutions until Tuesday, 16/10/2018, 18:00 into your assistant's box next to HG G 53.2.

Exercise 4.1 Let (Ω, \mathcal{F}) be a measurable space endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$. Recall that a *stopping time* is a random variable $\tau : \Omega \rightarrow \{0, 1, \dots, T\}$ with the property that

$$\{\tau \leq k\} \in \mathcal{F}_k$$

for $k = 0, 1, \dots, T$. Recall also the convention that $\inf \emptyset = +\infty$. If $X = (X_k)_{k=0,1,\dots,T}$ is an \mathbb{F} -adapted process and $B \in \mathcal{B}(\mathbb{R})$ a Borel set, then

$$\tau_{X,B} := \inf\{k = 0, 1, \dots, T : X_k \in B\}$$

is called the *first hitting time* of X on B .

- (a) Show that $\tau_{X,B} \wedge T$ is a stopping time.
- (b) Let τ be any stopping time. Show that there exist an adapted process X and a set $B \in \mathcal{B}(\mathbb{R})$ such that $\tau = \tau_{X,B}$. In other words, show that (up to truncating at T) every (first) hitting time of some adapted process X on some $B \in \mathcal{B}(\mathbb{R})$ is a stopping time and vice versa.
Hint: Try to construct such a process explicitly. It will depend on τ .

Exercise 4.2 Let $(\tilde{S}^0, \tilde{S}^1)$ be a *binomial model* and assume that $T = 1$, $u > r > 0$ and $-1 < d < 0$. For $\tilde{K} > 0$, define the functions $C(\cdot, \tilde{K})$ and $P(\cdot, \tilde{K}) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$C(x, \tilde{K}) := (x - \tilde{K})^+ := \max(0, x - \tilde{K}) \quad \text{and} \quad P(x, \tilde{K}) := (\tilde{K} - x)^+ := \max(0, \tilde{K} - x).$$

In financial terms, $C(\cdot, \tilde{K})$ is the payoff function of a *European call option with strike \tilde{K}* , and $P(\cdot, \tilde{K})$ is the payoff function of a *European put option with strike \tilde{K}* .

- (a) Construct a self-financing strategy $\varphi^{C(\tilde{K})} \hat{=} (V_0^{C(\tilde{K})}, \vartheta^{C(\tilde{K})})$ such that

$$V_1(\varphi^{C(\tilde{K})}) = \frac{C(\tilde{S}_1^1, \tilde{K})}{1+r} \quad P\text{-a.s.}$$

Hint: The exercise reduces to solving two linear equations.

- (b) Construct a self-financing strategy $\varphi^{P(\tilde{K})} \hat{=} (V_0^{P(\tilde{K})}, \vartheta^{P(\tilde{K})})$ such that

$$V_1(\varphi^{P(\tilde{K})}) = \frac{P(\tilde{S}_1^1, \tilde{K})}{1+r} \quad P\text{-a.s.}$$

Hint: The exercise reduces to solving two linear equations.

- (c) Prove the *put-call parity*

$$V_0^{P(\tilde{K})} + S_0^1 = V_0^{C(\tilde{K})} + \frac{\tilde{K}}{1+r}. \quad (*)$$

Give an economic interpretation of (*).

- (d) Compute $\lim_{\tilde{K} \rightarrow \infty} V_0^{C(\tilde{K})}$, $\lim_{\tilde{K} \rightarrow 0} V_0^{C(\tilde{K})}$, $\lim_{\tilde{K} \rightarrow \infty} V_0^{P(\tilde{K})}$ and $\lim_{\tilde{K} \rightarrow 0} V_0^{P(\tilde{K})}$. Can you guess the result before doing the computations?

Exercise 4.3 Consider a financial market $(\tilde{S}^0, \tilde{S}^1)$ with time horizon $T = 1$ consisting of a bank account and one stock defined on a probability space (Ω, \mathcal{F}, P) . Assume that $\tilde{S}_0^0 = \tilde{S}_0^1 = 1$ and $\tilde{S}_1^1 = e^Y$, where $Y \sim \mathcal{N}(0, 1)$ under P . Finally, assume that $\tilde{S}_1^0 = e^r$ for a deterministic $r \in (0, 1/2)$ and consider the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1}$ given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 := \mathcal{F}$.

- (a) Consider the map $Q : \mathcal{F} \rightarrow \mathbb{R}$ given by $Q[A] := E[Z \mathbf{1}_A]$, where

$$Z := \exp\left(-\left(\frac{1}{2} - r\right)Y - \frac{\left(\frac{1}{2} - r\right)^2}{2}\right).$$

Show that Q is a probability measure and that it is equivalent to P .

Hint: You can use that for $X \sim \mathcal{N}(\mu, \sigma^2)$, one has $E[e^{\alpha X}] = \exp(\alpha\mu + \frac{1}{2}\alpha^2\sigma^2)$.

- (b) Show that Q is an equivalent martingale measure for S^1 , i.e. that S^1 is a martingale under Q .
Hint: In this setting, $E_Q[S_1^1] = E[ZS_1^1]$.

- (c) Consider again the (undiscounted) payoff $C(\tilde{S}_1^1, \tilde{K}) = (\tilde{S}_1^1 - \tilde{K})^+$ of a long position in a European call option with strike \tilde{K} . Compute

$$V_0^C := E_Q\left[\frac{C(\tilde{S}_1^1, \tilde{K})}{\tilde{S}_1^0}\right].$$

- (d) Consider an enlargement of the market given by $(\tilde{S}^0, \tilde{S}^1, \tilde{S}^2)$, where we set $\tilde{S}_0^2 := V_0^C$ and $\tilde{S}_1^2 := C(\tilde{S}_1^1, \tilde{K})$. Is this market free of arbitrage?