



## Mathematical Foundations for Finance

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### Exercise 4

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## Definition 1 (Arbitrage opportunity)

An *arbitrage opportunity* is an admissible self-financing strategy  $\varphi \hat{=} (0, \vartheta)$  with zero initial wealth, with  $V_T(\varphi) \geq 0$   $P$ -a.s. and with  $P[V_T(\varphi) > 0] > 0$ . The financial market  $(\Omega, \mathcal{F}, \mathbb{F}, P, S^0, S^1)$  or shortly  $S$  is called *arbitrage-free* if there exist no arbitrage opportunities. Sometimes one also says that  $S$  satisfies (NA).

- Admissible so that we exclude strategies that we would not be able to carry out anyway (such as the doubling strategy).
- Self-financing and with zero initial investment at time  $k = 0$  so that no external financing is needed.
- $V_T(\varphi) \geq 0$   $P$ -a.s. so that we do not lose money  $P$ -a.s.
- $P[V_T(\varphi) > 0] > 0$  so that we stand a chance of making a gain.

## Lemma 2

Let  $(\Omega, \mathcal{F}, \mathbb{F}, P, S^0, S)$ , or shortly  $S$ , with  $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$  be a financial market in finite discrete time. If there exists a probability measure  $Q \approx P$  on  $\mathcal{F}_T$  such that  $S$  is a  $Q$ -martingale, then the market  $S$  is arbitrage-free.

- Here it is starting to be clear why we work with discounted price processes. A similar result for undiscounted prices does not hold.
- Intuition: our (intuitive) definition of arbitrage requires investment into at least two assets at minimum at one point in time – arbitrage is connected to how prices move relative to each other.
- Note that it is in fact sufficient if there *exists an equivalent local martingale measure* (ELMM).
- This lemma actually holds for continuous time as well as infinite time horizon.
- The next big result shows that the converse holds as well.

## Theorem 3 (Fundamental theorem of asset pricing)

Let  $(\Omega, \mathcal{F}, \mathbb{F}, P, S^0, S)$ , or shortly  $S$ , with  $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$  be a financial market in finite discrete time. Then  $S$  is arbitrage-free if and only if there exists an EMM for  $S$ .

- If we denote  $\mathbb{P}_e(S)$  the set of all EMMs for  $S$ , then we can shortly write  $(NA) \Leftrightarrow \mathbb{P}_e(S) \neq \emptyset$ .
- In relation to the previous lemma, if  $S$  is a  $Q$ -martingale, then it is also a  $Q$ -local martingale.
- Unlike the previous lemma, this theorem does not in general hold in continuous time or infinite horizon.
- This helps us to express the rather complicated requirement of creating a model for a market that is free of arbitrage in terms of the simple notion of expectation.

## Corollary 4

*The multiplicative multinomial model with parameters  $y_1 < \dots < y_m$  and  $r$  is arbitrage-free if and only if  $y_1 < r < y_m$ .*

## Corollary 5

*The multiplicative binomial model with parameters  $u > d$  and  $r$  is arbitrage-free if and only if  $d < r < u$ . In that case the EMM for  $S$  is uniquely defined by*

$$q_u = Q[Y_k = 1 + u] = \frac{r - d}{u - d}.$$

This makes intuitive sense. If  $\tilde{S}^1$  grew faster than  $\tilde{S}^0$  in all states of the world, then we would simply sell arbitrary amount of  $\tilde{S}^0$  and invest all the proceedings to  $\tilde{S}^1$  and we would have an arbitrage strategy.

# Basic Financial Terms

This course is mainly about developing the theory required for pricing of financial derivatives. These will start to occur in the exercise sheets as well as in the lecture. While from mathematical perspective most of these instruments can solely be viewed as functions, it is good to understand why they take the forms that they take.

## Definition 6 (Financial derivative)

A *financial derivative* is an instrument whose value is at least partially derived from one or more *underlying securities*.

- This a very broad and not a very insightful definition – there will be examples.
- The underlying can vary a lot – interest rate, inflation, commodities, currencies, indices, stocks etc.
- In this course, we will deal exclusively with *equity derivatives*, for which the underlying are stocks, and more specifically with *options*.

# Basic Financial Terms

## Definition 7 (European call option)

A *European call option* is a financial derivative that gives its holder the right, but not the obligation to buy the underlying security  $\tilde{S}$  at the *maturity*  $T$  in the future for a fixed price  $K$ , called the *strike price*.

## Definition 8 (European put option)

A *European put option* is a financial derivative that gives its holder the right, but not the obligation to sell the underlying security  $\tilde{S}$  at the *maturity*  $T$  in the future for a fixed price  $K$ , called the *strike price*.

- Since a rational investor would never exercise his or her option if the profit is negative, the payoffs of these options at time  $T$  can easily be seen to be  $C(\omega) = \max\{0, \tilde{S}_T(\omega) - K\}$  and  $P(\omega) = \max\{0, K - \tilde{S}_T(\omega)\}$ .
- The knowledge of the payoff still does not tell us too much about the price in between the inception of the contract and the maturity.
- It seems reasonable to set the price so that there is no arbitrage, which suggests that the price should be positive at all times.

Thank you for your attention!