Mathematical Foundations for Finance

Exercise sheet 4

Exercise 4.1 Let $(\Omega, \mathcal{F})$ be a measurable space endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$. Recall that a *stopping time* is a random variable $\tau : \Omega \to \{0, 1, \dots, T\}$ with the property that

$$\{\tau \leq k\} \in \mathcal{F}_k$$

for $k = 0, 1, \dots, T$. Recall also the convention that $\inf \emptyset = +\infty$. If $X = (X_k)_{k=0,1,\dots,T}$ is an $\mathbb{F}$-adapted process and $B \in \mathcal{B}(\mathbb{R})$ a Borel set, then

$$\tau_{X,B} := \inf\{k = 0, 1, \dots, T : X_k \in B\}$$

is called the *first hitting time* of X on B.

(a) Show that $\tau_{X,B} \wedge T$ is a stopping time.

(b) Let τ be any stopping time. Show that there exist an adapted process X and a set $B \in \mathcal{B}(\mathbb{R})$ such that $\tau = \tau_{X,B}$. In other words, show that (up to truncating at T) every (first) hitting time of some adapted process X on some $B \in \mathcal{B}(\mathbb{R})$ is a stopping time and vice versa.
Hint: Try to construct such a process explicitly. It will depend on τ.

Solution 4.1

(a) Fix a $k \in \{0, 1, \dots, T\}$. For any $j \in \{0, 1, \dots, k\}$, X_j is $\mathcal{F}_j$-measurable because X is adapted, which means that $\{X_j \in B\} = \{\omega \in \Omega : X_j(\omega) \in B\} \in \mathcal{F}_j \subset \mathcal{F}_k$. Moreover, by definition of $\tau_{X,B}$, we have

$$\{\tau_{X,B} \leq k\} = \big\{X_j \in B \text{ for some } j \in \{0, 1, \dots, k\}\big\} = \bigcup_{j=0}^{k} \{X_j \in B\} \in \mathcal{F}_k$$

because $\mathcal{F}_k$ as a σ-algebra is closed under countable unions. $\tau_{X,B}$ can, however, attain the value of $+\infty$ and thus does not satisfy our definition of a stopping time. However, since $\tau_{X,B} \wedge T$ can only attain values in $\{0, 1, \dots, T\}$ and

$$\{\tau_{X,B} \wedge T \leq k\} = \begin{cases} \{\tau_{X,B} \leq k\} & \text{for } k < T \\ \Omega & \text{for } k = T, \end{cases}$$

we have that $\tau_{X,B} \wedge T$ indeed is a stopping time.

(b) Given a stopping time τ, we define

$$X_k := \mathbb{1}_{\{\tau > k\}}$$

for $k = 0, 1, \dots, T$ and set $X := (X_k)_{k=0,1,\dots,T}$. Since τ is a stopping time, $\{\tau \leq k\}$ (and therefore $\{\tau > k\} = \{\tau \leq k\}^c$) is in $\mathcal{F}_k$ for every $k = 0, 1, \dots, T$. This implies that X is adapted. Moreover, $X_k(\omega) = 1$ for $\tau(\omega) > k$ and $X_k(\omega) = 0$ for $\tau(\omega) \leq k$ so that

$$\tau(\omega) = \inf\big\{k = 0, \dots, T : X_k(\omega) \in \{0\}\big\}.$$

Therefrom we clearly see that $\tau = \tau_{X,\{0\}}$ for an adapted $X = (X_k)_{k=0,1,\dots,T}$ defined by $X_k = \mathbb{1}_{\{\tau > k\}}$. Note that $\tau \leq T$, so $X_T = 0$ and hence $\tau_{X,\{0\}} \leq T$.

Exercise 4.2 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a *binomial model* and assume that $T = 1$, $u > r > 0$ and $-1 < d < 0$. For $\widetilde{K} > 0$, define the functions $C(\bullet, \widetilde{K})$ and $P(\bullet, \widetilde{K}) : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$C(x, \widetilde{K}) := (x - \widetilde{K})^+ := \max(0, x - \widetilde{K}) \quad \text{and} \quad P(x, \widetilde{K}) := (\widetilde{K} - x)^+ := \max(0, \widetilde{K} - x).$$

In financial terms, $C(\bullet, \widetilde{K})$ is the payoff function of a *European call option with strike* $\widetilde{K}$, and $P(\bullet, \widetilde{K})$ is the payoff function of a *European put option with strike* $\widetilde{K}$.

(a) Construct a self-financing strategy $\varphi^{C(\widetilde{K})} \mathrel{\widehat{=}} (V_0^{C(\widetilde{K})}, \vartheta^{C(\widetilde{K})})$ such that

$$V_1(\varphi^{C(\widetilde{K})}) = \frac{C(\widetilde{S}_1^1, \widetilde{K})}{1+r} \quad P\text{-a.s.}$$

Hint: The exercise reduces to solving two linear equations.

(b) Construct a self-financing strategy $\varphi^{P(\widetilde{K})} \mathrel{\widehat{=}} (V_0^{P(\widetilde{K})}, \vartheta^{P(\widetilde{K})})$ such that

$$V_1(\varphi^{P(\widetilde{K})}) = \frac{P(\widetilde{S}_1^1, \widetilde{K})}{1+r} \quad P\text{-a.s.}$$

Hint: The exercise reduces to solving two linear equations.

(c) Prove the *put-call parity*

$$V_0^{P(\widetilde{K})} + S_0^1 = V_0^{C(\widetilde{K})} + \frac{\widetilde{K}}{1+r}. \tag{$*$}$$

Give an economic interpretation of $(*)$.

(d) Compute $\lim_{\widetilde{K}\to\infty} V_0^{C(\widetilde{K})}$, $\lim_{\widetilde{K}\to 0} V_0^{C(\widetilde{K})}$, $\lim_{\widetilde{K}\to\infty} V_0^{P(\widetilde{K})}$ and $\lim_{\widetilde{K}\to 0} V_0^{P(\widetilde{K})}$. Can you guess the result before doing the computations?

Solution 4.2

(a) A self-financing strategy $\varphi^{C(\widetilde{K})} \mathrel{\widehat{=}} (V_0^{C(\widetilde{K})}, \vartheta^{C(\widetilde{K})})$ satisfies

$$V_1(\varphi^{C(\widetilde{K})}) = \frac{C(\widetilde{S}^1, \widetilde{K})}{1+r} \quad P\text{-a.s.}$$

if and only if we have

$$V_0^{C(\widetilde{K})} + \vartheta_1^{C(\widetilde{K})} \Delta S_1^1 = \frac{C(\widetilde{S^1}, \widetilde{K})}{1+r} \quad P\text{-a.s.}$$

Since $\vartheta_1^{C(\widetilde{K})}$ is $\mathcal{F}_0$-measurable, hence a constant, and S^1 only takes two values, the latter condition is equivalent to

$$\begin{aligned} V_0^{C(\widetilde{K})} + \vartheta_1^{C(\widetilde{K})} \frac{u-r}{1+r} S_0^1 &= \frac{C((1+u)S_0^1, \widetilde{K})}{1+r} \quad \text{and} \\ V_0^{C(\widetilde{K})} + \vartheta_1^{C(\widetilde{K})} \frac{d-r}{1+r} S_0^1 &= \frac{C((1+d)S_0^1, \widetilde{K})}{1+r}. \end{aligned} \tag{1}$$

Subtracting the two equations, multiplying by $(1+r)$ and dividing by S_0^1 yields

$$\begin{aligned} & \vartheta_1^{C(\widetilde{K})}(u-d) = C(1+u, \widetilde{K}/S_0^1) - C(1+d, \widetilde{K}/S_0^1) \\ \Leftrightarrow \quad & \vartheta_1^{C(\widetilde{K})} = \frac{C(1+u, \widetilde{K}/S_0^1) - C(1+d, \widetilde{K}/S_0^1)}{u-d}. \end{aligned}$$

Plugging this into (1) yields after rearranging

$$\begin{aligned} V_0^{C(\widetilde{K})} &= \frac{S_0^1}{(1+r)(u-d)}\Big((u-d)C(1+u,\widetilde{K}/S_0^1) \\ &\qquad\qquad - (u-r)\big(C(1+u,\widetilde{K}/S_0^1) - C(1+d,\widetilde{K}/S_0^1)\big)\Big) \\ &= \frac{S_0^1}{(1+r)(u-d)}\Big((r-d)C(1+u,\widetilde{K}/S_0^1) + (u-r)C(1+d,\widetilde{K}/S_0^1)\Big) \\ &= S_0^1\left(\frac{r-d}{u-d}\frac{C(1+u,\widetilde{K}/S_0^1)}{1+r} + \frac{u-r}{u-d}\frac{C(1+d,\widetilde{K}/S_0^1)}{1+r}\right) \\ &= \frac{r-d}{u-d}\frac{C\big(S_0^1(1+u),\widetilde{K}\big)}{1+r} + \frac{u-r}{u-d}\frac{C\big(S_0^1(1+d),\widetilde{K}\big)}{1+r}. \end{aligned}$$

This can also be written as $V_0^{C(\widetilde{K})} = E^*\left[\frac{C(\widetilde{S}_1^1,\widetilde{K})}{1+r}\right]$, where

$$P^*\left[\frac{\widetilde{S}_1^1}{S_0^1} = 1+u\right] = p^* := \frac{r-d}{u-d} \quad \text{and} \quad P^*\left[\frac{\widetilde{S}_1^1}{S_0^1} = 1+d\right] = 1-p^* := \frac{u-r}{u-d}.$$

(b) The same calculations as in (a) yield

$$\begin{aligned} \vartheta_1^{P(\widetilde{K})} &= \frac{P(1+u,\widetilde{K}/S_0^1) - P(1+d,\widetilde{K}/S_0^1)}{u-d}, \\ V_0^{P(\widetilde{K})} &= S_0^1\left(\frac{r-d}{u-d}\frac{P(1+u,\widetilde{K}/S_0^1)}{(1+r)} + \frac{u-r}{u-d}\frac{P(1+d,\widetilde{K}/S_0^1)}{1+r}\right) = E^*\left[\frac{P(\widetilde{S}_1^1,\widetilde{K})}{1+r}\right]. \end{aligned}$$

(c) For fixed $\widetilde{K} > 0$, by distinguishing the two cases $x \geq \widetilde{K}$ and $x < \widetilde{K}$, we easily get

$$C(x,\widetilde{K}) - P(x,\widetilde{K}) = x - \widetilde{K}.$$

Alternatively, $(\widetilde{K}-x)^+ = (x-\widetilde{K})^-$ gives

$$x - \widetilde{K} = (x-\widetilde{K})^+ - (x-\widetilde{K})^- = C(x,\widetilde{K}) - P(x,\widetilde{K}).$$

Thus, using the formulas for $V_0^{C(\widetilde{K})}$ and $V_0^{P(\widetilde{K})}$, we get

$$\begin{aligned} V_0^{C(\widetilde{K})} - V_0^{P(\widetilde{K})} &= \frac{S_0^1}{1+r}\left(\frac{r-d}{u-d}(1+u-\widetilde{K}/S_0^1) + \frac{u-r}{u-d}(1+d-\widetilde{K}/S_0^1)\right) \\ &= \frac{S_0^1}{1+r}((1-\widetilde{K}/S_0^1)+r) = S_0^1 - \frac{\widetilde{K}}{1+r}. \end{aligned} \tag{2}$$

Alternatively, we can get this by writing

$$V_0^{C(\widetilde{K})} - V_0^{P(\widetilde{K})} = E^*\left[\frac{C(\widetilde{S}_1^1,\widetilde{K}) - P(\widetilde{S}_1^1,\widetilde{K})}{1+r}\right] = E^*\left[\frac{\widetilde{S}_1^1}{1+r}\right] - \frac{\widetilde{K}}{1+r} = S_0^1 - \frac{\widetilde{K}}{1+r}.$$

Rearranging yields $(*)$. The economic interpretation of $(*)$ is that buying a stock and a put option with strike $\widetilde{K}$ and maturity T is equivalent to buying a call option with the same strike and the same maturity and a zero-coupon bond with the same maturity and face value $\widetilde{K}$.
Note: Even though we have just proved the put-call parity for a specific model for the market $(\widetilde{S}^0,\widetilde{S}^0)$, namely the one-period binomial model, no-arbitrage arguments can be used to prove this relationship in a model-free setting.

(d) For a fixed $x \geq 0$, we clearly have

$$\lim_{\widetilde{K}\to\infty} C(x,\widetilde{K}) = 0 \quad \text{and} \quad \lim_{\widetilde{K}\to 0} C(x,\widetilde{K}) = x.$$

Therefore we have

$$\lim_{\widetilde{K}\to\infty} V_0^{C(\widetilde{K})} = S_0^1\left(\frac{r-d}{u-d}\frac{0}{1+r} + \frac{u-r}{u-d}\frac{0}{1+r}\right) = 0,$$
$$\lim_{\widetilde{K}\to 0} V_0^{C(\widetilde{K})} = S_0^1\left(\frac{r-d}{u-d}\frac{1+u}{1+r} + \frac{u-r}{u-d}\frac{1+d}{1+r}\right) = S_0^1.$$

Using the put-call parity from (2), we get

$$\lim_{\widetilde{K}\to\infty} V_0^{P(\widetilde{K})} = \lim_{\widetilde{K}\to\infty}\left(\frac{\widetilde{K}}{1+r} - S_0^1 + V_0^{C(\widetilde{K})}\right) = +\infty,$$
$$\lim_{\widetilde{K}\to 0} V_0^{P(\widetilde{K})} = \lim_{\widetilde{K}\to 0}\left(\frac{\widetilde{K}}{1+r} - S_0^1 + V_0^{C(\widetilde{K})}\right) = 0.$$

Exercise 4.3 Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ with time horizon $T = 1$ consisting of a bank account and one stock defined on a probability space $(\Omega, \mathcal{F}, P)$. Assume that $\widetilde{S}_0^0 = \widetilde{S}_0^1 = 1$ and $\widetilde{S}_1^1 = e^Y$, where $Y \sim \mathcal{N}(0,1)$ under P. Finally, assume that $\widetilde{S}_1^0 = e^r$ for a deterministic $r \in (0, 1/2)$ and consider the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1}$ given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 := \mathcal{F}$.

(a) Consider the map $Q : \mathcal{F} \to \mathbb{R}$ given by $Q[A] := E[Z\mathbb{1}_A]$, where

$$Z := \exp\left(-\left(\frac{1}{2} - r\right)Y - \frac{(\frac{1}{2}-r)^2}{2}\right).$$

Show that Q is a probability measure and that it is equivalent to P.
Hint: You can use that for $X \sim \mathcal{N}(\mu, \sigma^2)$, one has $E\left[e^{\alpha X}\right] = \exp(\alpha\mu + \frac{1}{2}\alpha^2\sigma^2)$.

(b) Show that Q is an equivalent martingale measure for S^1, i.e. that S^1 is a martingale under Q.
Hint: In this setting, $E_Q\left[S_1^1\right] = E\left[ZS_1^1\right]$.

(c) Consider again the (undiscounted) payoff $C(\widetilde{S}_1^1, \widetilde{K}) = (\widetilde{S}_1^1 - \widetilde{K})^+$ of a long position in a European call option with strike $\widetilde{K}$. Compute

$$V_0^C := E_Q\left[\frac{C(\widetilde{S}_1^1, \widetilde{K})}{\widetilde{S}_1^0}\right].$$

(d) Consider an enlargement of the market given by $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$, where we set $\widetilde{S}_0^2 := V_0^C$ and $\widetilde{S}_1^2 := C(\widetilde{S}_1^1, \widetilde{K})$. Is this market free of arbitrage?

Solution 4.3

(a) In order for Q to be a probability measure, we must have that

1. $Q[A] \in [0,1]$ for all $A \in \mathcal{F}$,
2. $Q[\Omega] = 1$,
3. $Q\left[\bigcup_{i=1}^\infty A_i\right] = \sum_{i=1}^\infty Q[A_i]$ for any disjoint family of sets $(A_i)_{i\in\mathbb{N}} \in \mathcal{F}$.

In order for Q to be equivalent to P, we must additionally have that $P[A] = 0 \iff Q[A] = 0$.

For 2, note that setting

$$W := -\Big(\frac{1}{2} - r\Big)Y - \frac{(\frac{1}{2} - r)^2}{2},$$
$$\mu_Z := -\frac{(\frac{1}{2} - r)^2}{2} \qquad \text{and} \qquad \sigma_Z := \frac{1}{2} - r > 0,$$

we can see that $W \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$ under P. Using the hint, we compute

$$E[Z] = E[e^W] = \exp(\mu_Z + \sigma_Z^2/2) = \exp\left(-\frac{(\frac{1}{2} - r)^2}{2} + \frac{(\frac{1}{2} - r)^2}{2}\right) = 1.$$

Next, we have that

$$0 \leq Z\mathbb{1}_A \leq Z \quad P\text{-a.s. for all } A \in \mathcal{F} \tag{3}$$

where the first inequality follows from the fact that $Z > 0$ P-a.s. (since the function $x \mapsto e^x$ is strictly positive) and the second one from the fact that every $A \in \mathcal{F}$ is a subset of Ω so $\mathbb{1}_A \leq \mathbb{1}_\Omega$ and we clearly have that $Z\mathbb{1}_\Omega = Z$. But (3) gives that

$$0 \leq E[Z\mathbb{1}_A] \leq E[Z] = 1,$$

which in turn gives 1.

For 3, consider a family of disjoint sets $(A_i)_{i\in\mathbb{N}} \in \mathcal{F}$. We have that

$$\sum_{i=1}^{\infty} Q[A_i] = \lim_{n\to\infty} \sum_{i=1}^{n} E[Z\mathbb{1}_{A_i}] = \lim_{n\to\infty} E\left[\sum_{i=1}^{n} Z\mathbb{1}_{A_i}\right]. \tag{4}$$

However, since the family of sets $(A_i)_{i\in\mathbb{N}}$ is disjoint, the random variable $\sum_{i=1}^n Z\mathbb{1}_{A_i}$ is simply equal to Z on $B = \cup_{i=1}^n A_i$ and to 0 on B^c. So for all $n \in \mathbb{N}$ we have that

$$\left|\sum_{i=1}^{n} Z\mathbb{1}_{A_i}\right| \leq Z \quad P\text{-a.s.}$$

Since, $Z \in L^1(P)$, we can use the dominated convergence theorem in (4) to obtain that

$$\sum_{i=1}^{\infty} Q[A_i] = E\left[\sum_{i=1}^{\infty} Z\mathbb{1}_{A_i}\right] = E[Z\mathbb{1}_{\cup_{i\in\mathbb{N}}A_i}] = Q\left[\bigcup_{i\in\mathbb{N}} A_i\right],$$

which gives 3.

For the equivalence of Q and P we use that $Z > 0$ P-a.s. which then means that

$$Q[A] = 0 \iff \mathbb{1}_A = 0 \ P\text{-a.s.} \iff E[\mathbb{1}_A] = 0 \iff P[A] = 0.$$

(b) In order to prove that Q is an EMM for S^1, we have to show that S^1 is a martingale under Q. Since adaptedness of a process does not depend on a probability measure, we only have to check that S_1^1 is integrable and that $E_Q[S_1^1 \,|\, \mathcal{F}_0] = E_Q[S_1^1] = E[ZS_1^1] = S_0^1$. We can rewrite

$$ZS_1^1 = \exp\left(-\left(\frac{1}{2} - r\right)Y - \frac{(\frac{1}{2} - r)^2}{2}\right)\exp(Y - r)$$

as $ZS_1^1 = e^{\widetilde{Y}}$ with

$$\widetilde{Y} = \left(\frac{1}{2} + r\right)Y - r - \frac{(\frac{1}{2} - r)^2}{2} = \left(\frac{1}{2} + r\right)Y - \frac{(r + \frac{1}{2})^2}{2}.$$

Since $Y \sim \mathcal{N}(0,1)$ under P, we also have that $\widetilde{Y} \sim \mathcal{N}(\mu, \sigma^2)$ under P with

$$\mu = -\frac{(r+\frac{1}{2})^2}{2} \qquad \text{and} \qquad \sigma^2 = (\frac{1}{2}+r)^2.$$

Using the hint, we obtain that

$$E_Q\left[|S_1^1|\right] = E_Q\left[S_1^1\right] = E\left[ZS_1^1\right] = E\left[e^{\widetilde{Y}}\right] = 1 = \widetilde{S}_0^1,$$

giving both the integrability of S_1^1 and the martingale property of S^1 under Q.

(c) Let's start by writing $C(\widetilde{S}_1^1, \widetilde{K}) = (\widetilde{S}_1^1 - \widetilde{K})\mathbb{1}_{\{\widetilde{S}_1^1 \geq \widetilde{K}\}} = (e^Y - \widetilde{K})\mathbb{1}_{\{Y \geq \log \widetilde{K}\}}$. Then

$$\begin{aligned}
E_Q\left[\frac{C(\widetilde{S}_1^1, \widetilde{K})}{\widetilde{S}_1^0}\right] &= e^{-r}E_Q[(e^Y - \widetilde{K})1_{\{Y \geq \log \widetilde{K}\}}] \\
&= e^{-r}E_P[Z(e^Y - \widetilde{K})1_{\{Y \geq \log \widetilde{K}\}}] \\
&= e^{-r}\int_{\log \widetilde{K}}^{\infty} (e^y - \widetilde{K}) \exp\left(-\left(\frac{1}{2}-r\right)y - \frac{\left(\frac{1}{2}-r\right)^2}{2}\right)\phi(y)dy \\
&= e^{-r}(I_1 + I_2),
\end{aligned}$$

where ϕ denotes the density of $\mathcal{N}(0,1)$. These integrals are calculated by completing squares as follows:

$$\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}}\int_{\log \widetilde{K}}^{\infty} \exp\left(\left(\frac{1}{2}+r\right)y - \frac{\left(\frac{1}{2}-r\right)^2}{2} - \frac{y^2}{2}\right)dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{\log \widetilde{K}}^{\infty} \exp\left(-\frac{\left(y - \left(\frac{1}{2}+r\right)\right)^2}{2} + r\right)dy \\
&= \frac{e^r}{\sqrt{2\pi}}\int_{\log \widetilde{K} - \left(\frac{1}{2}+r\right)}^{\infty} e^{-\frac{x^2}{2}}dx \\
&= e^r\left(1 - \Phi\left(\log \widetilde{K} - \left(\frac{1}{2}+r\right)\right)\right) = e^r\Phi\left(-\log \widetilde{K} + \left(\frac{1}{2}+r\right)\right).
\end{aligned}$$

In a similar fashion,

$$\begin{aligned}
I_2 &= -\frac{\widetilde{K}}{\sqrt{2\pi}}\int_{\log \widetilde{K}}^{\infty} \exp\left(-\left(\frac{1}{2}-r\right)y - \frac{\left(\frac{1}{2}-r\right)^2}{2} - \frac{y^2}{2}\right)dy \\
&= -\frac{\widetilde{K}}{\sqrt{2\pi}}\int_{\log \widetilde{K}}^{\infty} \exp\left(-\frac{\left(y - \left(r-\frac{1}{2}\right)\right)^2}{2}\right)dy \\
&= -\widetilde{K}\Phi\left(-\log \widetilde{K} + \left(r - \frac{1}{2}\right)\right).
\end{aligned}$$

Returning to the original expression yields

$$E_Q\left[\frac{C(\widetilde{S}_1^1, \widetilde{K})}{\widetilde{S}_1^0}\right] = \Phi\left(-\log \widetilde{K} + \left(r + \frac{1}{2}\right)\right) - \widetilde{K}e^{-r}\Phi\left(-\log \widetilde{K} + \left(r - \frac{1}{2}\right)\right).$$

(d) Yes, it is free of arbitrage, since Q, as can easily be verified using the fact t, is an equivalent martingale measure not only for S^1 but also for and S^2. Lemma II.1.2 in the lecture notes thus gives us that the desired result.