Mathematical Foundations for Finance

Exercise sheet 5

Please hand in your solutions until Tuesday, 23/10/2018, 18:00 into your assistant's box next to HG G 53.2.

Exercise 5.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1\ldots,T}$ and let $\mathcal{F}_0$ be trivial. Let $X = (X_k)_{k=0,1\ldots,T}$ be a local martingale and $\vartheta = (\vartheta_k)_{k=0,1,\ldots,T}$ a real-valued predictable process.

(a) Show that if X is bounded from below, then X is a supermartingale.
Hint: Fatou's lemma.

(b) Is the stochastic integral with respect to a supermartingale always a supermartingale? Why or why not?

Exercise 5.2 Consider on a probability space $(\Omega, \mathcal{F}, P)$ a random variable X which is uniformly distributed on $(0, 1)$. Let $Y = (Y_k)_{k=0,1,2}$ be the process given by

$$Y_0 = 0, \quad Y_1 = X - \frac{1}{2}, \quad \text{and} \quad Y_2 = X - \frac{1}{2} + \frac{B}{X^2}$$

for some random variable B independent of X and such that $P[B = 1] = P[B = -1] = 1/2$. Finally define the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ by $\mathcal{F}_k = \sigma(Y_i,\ i \leq k)$.

(a) Prove that Y is not a martingale.
Hint: There is an integrability issue.

(b) Consider the sequence $(\tau_n)_{n\in\mathbb{N}}$ given by $\tau_n := \mathbb{1}_{\{X\geq 1/n\}} + 1$. Show that it forms a sequence of stopping times increasing to 2 with $P[\tau_n = 2] \to 1$ as $n \to \infty$.

(c) Prove that Y is a local martingale by showing that $(\tau_n)_{n\in\mathbb{N}}$ can be chosen as localizing sequence.

Exercise 5.3 We say that the market $(\Omega, \mathcal{F}, \mathbb{F}, P, S^0, S^1)$, or shortly just S, satisfies (NA') if there exist no self-financing strategies $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with zero initial wealth (including non-admissible ones) such that $V_T(\varphi) \geq 0$ P-a.s. and $P[V_T(\varphi) > 0] > 0$. This is like (NA) except that we drop the requirement of admissibility of $\varphi \mathrel{\widehat{=}} (0, \vartheta)$. Prove that $\neg(NA') \implies \neg(NA)$ (the contraposition of $(NA) \implies (NA')$) using the steps below.

(a) First show that if we restrict ourselves to the class of self-financing (not necessarily admissible strategies) with $G(\vartheta) \geq 0$, then we indeed have $\neg(NA') \implies \neg(NA)$.

(b) Now suppose that we have a strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ such that $P[G_k(\vartheta) < 0] > 0$ for some $k \in \{1, \ldots, T\}$. Modify the strategy φ appropriately so that $G(\vartheta) \geq 0$. This puts us into the setting of (a) and concludes the proof.

(c) Explain why this also gives us that $(NA) \implies (NA')$.