ETH zürich

Mathematical Foundations for Finance

Exercise 5

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Some Concepts from Probability Theory

Lemma 1 (Fatou's lemma)

Let X_n be a sequence of random variables on $(\Omega, \mathcal{F}, P)$ with $X_n \geq 0$ P-a.s. Then we have that

$$E_P\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} E_P\left[X_n\right].$$

Lemma 2 (Fatou's lemma for conditional expectation)

Let X_n be a sequence of random variables on $(\Omega, \mathcal{F}, P)$ with $X_n \geq 0$ P-a.s. and let $\mathcal{G} \subseteq \mathcal{F}$. Then

$$E_P\left[\liminf_{n\to\infty} X_n \,\middle|\, \mathcal{G}\right] \leq \liminf_{n\to\infty} E_P\left[X_n \,|\, \mathcal{G}\right] \quad P\text{-a.s.}$$

- The assumptions on X_n are weaker than for the monotone convergence and dominated convergence theorem (see the slides for week 1).
- Can be generalized to sequences of random variables that are bounded below, which is what you will need to use in this week's exercise sheet.

Theorem 3 (Radon–Nikodým theorem)

Let $(\Omega, \mathcal{F})$ be a measurable space. If Q is a σ-finite measure on $(\Omega, \mathcal{F})$ which is absolutely continuous with respect to a σ-finite measure P on $(\Omega, \mathcal{F})$, then there exists a measurable function $\mathcal{D} \geq 0$ P-a.s., such that for $A \in \mathcal{F}$ we have that

$$Q[A] = E_P[\mathcal{D}\mathbb{1}_A] = \int_A \mathcal{D} dP.$$

If we even have that $Q \approx P$, then $D > 0$ P-a.s.

- We are working with probability measures on $(\Omega, \mathcal{F})$, which are finite measures (i.e. $P[\Omega] = 1 < \infty$) and therefore σ-finite.
- The random variable $\mathcal{D}$ is often denoted by $\frac{dQ}{dP}$ and called the *Radon-Nikodým derivative* of Q with respect to P.
- This is the theorem that we implicitly use when computing the expectation of random variables.

Some Concepts from Probability Theory

The process $Z = (Z_k)_{k=0,1,\dots,T}$ defined by

$$Z_k := E_P\left[\frac{dQ}{dP}\,\middle|\,\mathcal{F}_k\right]$$

for some filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ is called the *density process* of Q with respect to P.

- Radon-Nikodým gives us that $Z \geq 0$ (and $Z > 0$ if $Q \approx P$) P-a.s.
- We have that $E[Z_k] = 1$ (see the following point).
- The density process is a P-martingale:
  - Adaptedness clear by the definition of conditional expectation.
  - $E_P\left[Z_k\right] = E_P\left[E_P\left[\frac{dQ}{dP}\,\middle|\,\mathcal{F}_k\right]\right] = E_P\left[\frac{dQ}{dP}\right] = E_Q\left[\mathbb{1}_\Omega\right] = Q[\Omega] = 1 \implies Z \in L^1(Q)$.
  - $E_P\left[Z_k\,|\,\mathcal{F}_j\right] = E_P\left[E_P\left[\frac{dQ}{dP}\,\middle|\,\mathcal{F}_k\right]\,\middle|\,\mathcal{F}_j\right] = E_P\left[\frac{dQ}{dP}\,\middle|\,\mathcal{F}_j\right] = Z_j$.

Attainable Payoffs

Definition 4 (Attainable payoff)

A payoff $H \in L^0_+(\mathcal{F}_T)$ is called *attainable* if there exists an admissible self-financing strategy $\varphi = (V_0, \vartheta)$ with $V_T(\varphi) = H$ P-a.s.

- The strategy φ from the previous definition is said to replicate H and is called a *replicating strategy* for H.
- It is a priori clear that attainable payoffs exist – take any admissible self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ and define $H = V_0 + \vartheta \bullet S_T$. This payoff will be replicable with $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ being the replicating strategy for H.

Valuation of Attainable Payoffs

Attainable payoffs are easy to price, since they must at every point in time have the same value as the replicating strategy, otherwise there is arbitrage.

Theorem 5

Consider a financial market model $(\Omega, \mathcal{F}, \mathbb{F}, P, S^0, S)$ *in finite discrete time and suppose that* S *is arbitrage-free and* $\mathcal{F}_0$ *in* $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ *is trivial. Then every attainable payoff* $H \in L^0_+(\mathcal{F}_T)$ *has a unique price process* $V^H = (V^H_k)_{k=0,1,\dots,T}$*, which admits no arbitrage. It is given by*

$$V^H_k = E_Q\left[H \,\middle|\, \mathcal{F}_k\right] = V_k(V_0, \vartheta) \quad \text{for } k = 0, 1, \dots, T$$

for any EMM $Q \in \mathbb{P}_e(S)$ *and for any replicating strategy* $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ *for* H.

A stylized approach to how to price an arbitrary attainable payoff can be found on page 50 in the lecture notes.

Non-attainable Payoffs

Problem

It might not be completely clear whether non-attainable payoffs even exists.

- For a payoff to be attainable, we need an admissible self-financing strategy whose terminal value is equal to the payoff in each state of the world (up to null sets).
- In a finite probability space, the question of existence of such a strategy translates to a question of existence of a solution to a linear system.
- It is easier to imagine that a linear system with no solution can occur.

Complete and Incomplete Markets

Definition 6 (Complete market)

A financial market model is called *complete* if every payoff $H \in L^0_+(\mathcal{F}_T)$ is attainable.

Definition 7 (Incomplete market)

A financial market model is called *incomplete* if there exists a payoff $H \in L^0_+(\mathcal{F}_T)$ which is not attainable.

Even though the definitions can be easily combined, we explicitly include the definition of an incomplete market to make it clear that there can also be attainable payoffs in an incomplete market.

The stylized approach to pricing attainable payoffs can be used in both complete and incomplete markets.

Is there an easy way to distinguish between complete and incomplete markets in finite discrete time?

Theorem 8 (Second fundamental theorem of asset pricing)

Consider a financial market model in finite discrete time and assume that S is arbitrage-free, $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. Then S is complete if and only if there is a unique equivalent martingale measure for S. In brief,

$$(NA) + completeness \iff P_e(S) \text{ is a singleton}.$$

- This gives a simple way how to verify whether our market is complete or not.
- We have already seen that there exists a unique EMM in the binomial model – all payoffs are attainable.
- We have also seen that there is infinite number of EMMs for the multinomial model – there exist non-attainable payoffs.

Problems in Incomplete Markets

There are number of unanswered questions and problems with incomplete markets:

1. If we know that our market model is incomplete, is there a simple way to distinguish an attainable payoff from a non-attainable one?

Theorem 9 (Characterization of attainable payoffs)

Consider a financial market in finite discrete time and suppose S is arbitrage-free and $\mathcal{F}_0$ is trivial. For any payoff $H \in L^0_+(\mathcal{F}_T)$ the following are equivalent:

- *H is attainable.*
- $\sup_{Q \in P_e(S)} E_Q[H] < \infty$ *is attained for some* $Q^* \in P_e(S)$.
- *The mapping* $P_e(S) \to \mathbb{R}$, $Q \mapsto E_Q[H]$ *is constant.*

Problems in Incomplete Markets

2. Pricing, by replication is not possible anymore, but there are infinitely many ways to assign a price process to a non-attainable payoff $H \in L_0^+(\mathcal{F}_T)$ so that the extended market is arbitrage-free (conditional expectation under any EMM). Since every investor can pick his own arbitrage-free price process, what should be the prevailing price in the market?

   Answer: We still want to keep (NA) *– pick a unique EMM by imposing additional conditions (behavior or preferences of market participants; convenience).*

3. If there are infinitely many ways to assign a price process to $H \in L_0^+(\mathcal{F}_T)$, how do we hedge?

   Answer: We often have to resort to (partial) hedging using more assets.

Thank you for your attention!