Mathematical Foundations for Finance

Exercise sheet 5

Exercise 5.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1\dots,T}$ and let $\mathcal{F}_0$ be trivial. Let $X = (X_k)_{k=0,1\dots,T}$ be a local martingale and $\vartheta = (\vartheta_k)_{k=0,1,\dots,T}$ a real-valued predictable process.

(a) Show that if X is bounded from below, then X is a supermartingale.
Hint: Fatou's lemma.

(b) Is the stochastic integral with respect to a supermartingale always a supermartingale? Why or why not?

Solution 5.1

(a) By the definition of a local martingale, there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to T such that the stopped process $X^{\tau_n} = (X_{\tau_n \wedge k})_{k=0,1\dots,T}$ is a martingale for each $n \in \mathbb{N}$. By the martingale property, we have

$$E\left[X_{\tau_n\wedge k} \,|\, \mathcal{F}_j\right] = X_{\tau_n \wedge j}$$

for every $0 \le j \le k \le T$. Since we clearly have that for all $j = 0, 1, \dots, T$, $\lim_{n\to\infty} X_{\tau_n\wedge j} = X_j$ P-a.s. and X is bounded below, Fatou's lemma gives us that

$$E\left[X_k \,|\, \mathcal{F}_j\right] \le \liminf_{n\to\infty} E\left[X_{\tau_n\wedge k} \,|\, \mathcal{F}_j\right] = \liminf_{n\to\infty} X_{\tau_n\wedge j} = X_j, \tag{1}$$

which is the supermartingale property. Adaptedness of X is clear from the fact that X is a local martingale by assumption. For integrability, using again the result from (1), we have

$$E\left[X_k\right] = E\left[X_k \,|\, \mathcal{F}_0\right] \le X_0 \in \mathbb{R},$$

where for the first equality we use the fact that $\mathcal{F}_0$ is trivial and then the supermartingale property that we have shown already. Since X is bounded from below by assumption we also have that $E\left[|X_k|\right] < \infty$ for all $0 \le k \le T$. So X is indeed a supermartingale.

(b) In order to conclude that this statement is in general not true it is enough to consider the trivially predictable process $\vartheta \equiv -1$. In that case we obtain for a supermartingale X that

$$E\left[(\vartheta \bullet X)_k - (\vartheta \bullet X)_{k-1} \,|\, \mathcal{F}_{k-1}\right] = E\left[-(X_k - X_{k-1}) \,|\, \mathcal{F}_{k-1}\right] = -E\left[X_k - X_{k-1} \,|\, \mathcal{F}_{k-1}\right] \ge 0,$$

since $E\left[X_k - X_{k-1} \,|\, \mathcal{F}_{k-1}\right] \le 0$ by assumption. The above, however, implies that $\vartheta \bullet X$ is a submartingale since integrability and adaptedness of $\vartheta \bullet X$ are clear. So if X is not a martingale (in which case it would be both a supermartingale and a submartingale), $\vartheta \bullet X$ is not a supermartingale.

If we assume ϑ to be additionally positive and bounded, then the stochastic integral process can be shown to be a supermartingale using the same approach as in Exercise 3.1 (b).

Exercise 5.2 Consider on a probability space $(\Omega, \mathcal{F}, P)$ a random variable X which is uniformly distributed on $(0, 1)$. Let $Y = (Y_k)_{k=0,1,2}$ be the process given by

$$Y_0 = 0, \quad Y_1 = X - \frac{1}{2}, \quad \text{and} \quad Y_2 = X - \frac{1}{2} + \frac{B}{X^2}$$

for some random variable B independent of X and such that $P[B = 1] = P[B = -1] = 1/2$. Finally define the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ by $\mathcal{F}_k = \sigma(Y_i,\ i \le k)$.

(a) Prove that Y is not a martingale.
Hint: There is an integrability issue.

(b) Consider the sequence $(\tau_n)_{n\in\mathbb{N}}$ given by $\tau_n := \mathbb{1}_{\{X\geq 1/n\}} + 1$. Show that it forms a sequence of stopping times increasing to 2 with $P[\tau_n = 2] \to 1$ as $n \to \infty$.

(c) Prove that Y is a local martingale by showing that $(\tau_n)_{n\in\mathbb{N}}$ can be chosen as localizing sequence.

Solution 5.2

(a) First note that

$$E\left[\left|\frac{1}{X^2}\right|\right] = E\left[\frac{1}{X^2}\right] = \int_0^1 \frac{1}{x^2}\, dx = \infty.$$

By the triangle inequality, we get $|Y_2| \geq \left|\frac{1}{X^2}\right| - \left|X - \frac{1}{2}\right|$ and hence

$$E[|Y_2|] \geq E\left[\left|\frac{1}{X^2}\right|\right] - E\left[\left|X - \frac{1}{2}\right|\right] = \infty,$$

which shows that Y_2 is not integrable and thus that Y is not a martingale.

(b) We first check that τ_n is a stopping time for each $n \in \mathbb{N}$. Fix an $n \in \mathbb{N}$ and note that since $\tau_n \geq 1$, we already have that $\{\tau_n \leq 0\} = \emptyset \in \mathcal{F}_0$. Next note that X is $\mathcal{F}_1$-measurable since $X = Y_1 + 1/2$, so that we also get $\{X < 1/n\} \in \mathcal{F}_1$. Hence we can write

$$\{\tau_n \leq 1\} = \{X < 1/n\} \in \mathcal{F}_1 \qquad \text{and} \qquad \{\tau_n \leq 2\} = \Omega \in \mathcal{F}_2,$$

and thus conclude that τ_n is a stopping time for all $n \in \mathbb{N}$.

Since $\mathbb{1}_{\{X\geq 1/n\}} \leq \mathbb{1}_{\{X\geq 1/(n+1)\}}$, we already see that $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence. Moreover, noting that for each $\omega \in \Omega$ there exists an $n \in \mathbb{N}$ such that $X(\omega) \geq 1/n$, we also have that $\lim_{n\to\infty} \tau_n = 2$ P-a.s., and we can thus conclude that $(\tau_n)_{n\in\mathbb{N}}$ is increasing to 2. Finally, observe that $P[\tau_n = 2] = P[X \geq 1/n] = 1 - 1/n \to 1$ for $n \to \infty$.

(c) In order to prove that Y is a local martingale, we show that $Y^{\tau_n} = (Y_{k\wedge\tau_n})_{k=0,1,2}$ is a martingale. Fix an $n \in \mathbb{N}$ and note that $Y_0^{\tau_n} = Y_0 = 0$,

$$Y_1^{\tau_n} = Y_1 = X - \frac{1}{2}, \quad \text{and} \quad Y_2^{\tau_n} = Y_1\mathbb{1}_{\{X<1/n\}} + Y_2\mathbb{1}_{\{X\geq 1/n\}} = X - \frac{1}{2} + \frac{B}{X^2}\mathbb{1}_{\{X\geq 1/n\}}.$$

One can easily see that the process Y^{τ_n} is adapted and that $Y_0^{\tau_n}$ and $Y_1^{\tau_n}$ are integrable, since X is bounded. Moreover, using again the triangle inequality, we can compute

$$E[|Y_2^{\tau_n}|] \leq E\left[\left|X - \frac{1}{2}\right|\right] + E\left[\frac{1}{X^2}\mathbb{1}_{\{X\geq 1/n\}}\right] \leq \frac{1}{4} + n^2 < \infty,$$

and thus conclude that $Y_k^{\tau_n}$ is integrable for all $k \in \{0, 1, 2\}$.

For the martingale condition, first note that $E[X] = 1/2$ and hence

$$E[Y_1^{\tau_n} \,|\, \mathcal{F}_0] = E[X] - 1/2 = 0 = Y_0^{\tau_n}.$$

Moreover, using again that X is $\mathcal{F}_1$-measurable, we also have that

$$E[Y_2^{\tau_n} \,|\, \mathcal{F}_1] = E\left[X - \frac{1}{2} + \frac{B}{X^2}\mathbb{1}_{\{X\geq 1/n\}} \,\middle|\, \mathcal{F}_1\right] = X - \frac{1}{2} + E[B \,|\, \mathcal{F}_1]\frac{1}{X^2}\mathbb{1}_{\{X\geq 1/n\}} \quad P\text{-a.s.}$$

Noting that since B is independent of X, it is also independent of $\mathcal{F}_1$, we can deduce that

$$E[Y_2^{\tau_n} \,|\, \mathcal{F}_1] = X - \frac{1}{2} + E[B]\frac{1}{X^2}\mathbb{1}_{\{X\geq 1/n\}} = X - \frac{1}{2} = Y_1^{\tau_n} \quad P\text{-a.s.}$$

We can thus conclude that Y is a local martingale with localizing sequence $(\tau_n)_{n\in\mathbb{N}}$.

Exercise 5.3 We say that the market $(\Omega, \mathcal{F}, \mathbb{F}, P, S^0, S^1)$, or shortly just S, satisfies (NA') if there exist no self-financing strategies $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with zero initial wealth (including non-admissible ones) such that $V_T(\varphi) \geq 0$ P-a.s. and $P[V_T(\varphi) > 0] > 0$. This is like (NA) except that we drop the requirement of admissibility of $\varphi \mathrel{\widehat{=}} (0, \vartheta)$. Prove that $\neg(NA') \implies \neg(NA)$ (the contraposition of $(NA) \implies (NA')$) using the steps below.

(a) First show that if we restrict ourselves to the class of self-financing (not necessarily admissible strategies) with $G(\vartheta) \geq 0$, then we indeed have $\neg(NA') \implies \neg(NA)$.

(b) Now suppose that we have a strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ such that $P[G_k(\vartheta) < 0] > 0$ for some $k \in \{1, \dots, T\}$. Modify the strategy φ appropriately so that $G(\vartheta) \geq 0$. This puts us into the setting of (a) and concludes the proof.

(c) Explain why this also gives us that $(NA) \implies (NA')$.

Solution 5.3

(a) Assume that there exists a self-financing strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with zero initial wealth, $V_T(\varphi) \geq 0$ P-a.s., $P[V_T(\varphi) > 0] > 0$ and $G(\vartheta) \geq 0$ P-a.s. In other words, φ is an arbitrage strategy in the sense of (NA'). Since our strategy is self-financing and with zero initial wealth by assumption, we have that $V_k(\varphi) = G_k(\varphi)$ P-a.s. for all $k = 0, 1, \dots, T$. Since $G(\vartheta) \geq 0$ P-a.s., this also means that $V(\varphi) \geq 0$ P-a.s., which in turns means that φ is 0-admissible and therefore admissible. This implies that φ is an arbitrage opportunity in the sense of (NA) as well.

(b) We know that if $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ is an arbitrage opportunity in the sense of (NA'), then we have that $0 \leq V_T(\varphi) = G_T(\vartheta)$ P-a.s. We must therefore have that $P[G_k(\vartheta) < 0] > 0$ for some $k \in \{1, \dots, T-1\}$.

The idea of how to modify the strategy ϑ is simple – make sure that we are out of the market at times when when there is a chance that $G_k(\vartheta) < 0$. More formally, define

$$\begin{aligned} k_0 &:= \max\{k : P[G_k(\vartheta) < 0] > 0\}, \\ A &:= \{G_{k_0}(\vartheta) < 0\}, \\ \vartheta'_k &:= \begin{cases} 0 & \text{if } k \leq k_0, \\ \vartheta_k \mathbb{1}_A & \text{if } k > k_0. \end{cases} \end{aligned}$$

Note that k_0 is not random. For $k \leq k_0$ we have that $\vartheta'_k = 0$, which is $\mathcal{F}_0$-measurable for all $k \leq k_0$ and thus also $\mathcal{F}_{k-1}$-measurable. For $k > k_0$ we have that $A \in \mathcal{F}_{k_0} \subseteq \mathcal{F}_{k-1}$ and since ϑ_k is $\mathcal{F}_{k-1}$-measurable by assumption, we have that $\vartheta_k \mathbb{1}_A$ is $\mathcal{F}_{k-1}$-measurable too. This means that ϑ is predictable and induces the self-financing strategy $\varphi' \mathrel{\widehat{=}} (0, \vartheta')$.

What remains to be checked is that our strategy is indeed 0-admissible. We have that

$$V_k(\varphi') = \sum_{j=1}^{k} \vartheta'_j \Delta S^1_j = \begin{cases} 0 & \text{if } k \leq k_0, \\ \big(G_k(\vartheta) - G_{k_0}(\vartheta)\big)\mathbb{1}_A & \text{if } k > k_0. \end{cases}$$

Since $G_k(\vartheta) \geq 0$ P-a.s. for all $k > k_0$ and $G_{k_0}(\vartheta) < 0$ for all $\omega \in A$, we have that $\big(G_k(\vartheta) - G_{k_0}(\vartheta)\big)\mathbb{1}_A \geq 0$ P-a.s. and thus also that $V(\varphi') \geq 0$ P-a.s. This gives the 0-admissibility of φ'. But we clearly have that $P[A] > 0$ since we assume that $P[G_k(\vartheta) < 0] > 0$ for some $k \in \{0, 1, \dots, T\}$, which means that φ is also an arbitrage opportunity.

As we can in this way modify every arbitrage strategy in the sense of (NA') into an arbitrage strategy in the sense of (NA), this concludes the proof.

(c) This works because a conditional statement of the form $A \implies B$ and its contraposition are logically equivalent. Knowing that the contraposition holds true, we could easily obtain a contradiction to (NA) if we assumed that $(NA) \not\Longrightarrow (NA')$.