Mathematical Foundations for Finance

Exercise sheet 7

Exercise 7.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$, and let τ be an $\mathbb{F}$-stopping time. We define

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq k\} \in \mathcal{F}_k \text{ for all } k = 0, 1, \dots, T\}.$$

(a) Show that $\mathcal{F}_\tau$ is a σ-algebra.

(b) Show that if we set $\tau \equiv k_0$ for a fixed $k_0 \in \{0, 1, \dots, T\}$, we have that $\mathcal{F}_\tau = \mathcal{F}_{k_0}$.

(c) Show that for a random variable $Y \in L^0_+(\mathcal{F})$, we have that

$$E[Y \,|\, \mathcal{F}_\tau]\, \mathbb{1}_{\{\tau=k\}} = E[Y \,|\, \mathcal{F}_k]\, \mathbb{1}_{\{\tau=k\}} \text{ } P\text{-a.s. for all } k \in \{0, 1, \dots, T\},$$

i.e. that $E[Y \,|\, \mathcal{F}_\tau] = E[Y \,|\, \mathcal{F}_k]$ P-a.s. on the set $\{\tau = k\}$ or, equivalently,

$$E[Y \,|\, \mathcal{F}_\tau] = \sum_{k=0}^{T} \mathbb{1}_{\{\tau=k\}} E[Y \,|\, \mathcal{F}_k] \;\; P\text{-a.s.}$$

Solution 7.1

(a) By the definition of a σ-algebra, we need to verify that $\Omega \in \mathcal{F}_\tau$, that if $A \in \mathcal{F}_\tau$, then also $A^c \in \mathcal{F}_\tau$, and that if $(A_n)_{n\in\mathbb{N}}$ is a sequence of sets in $\mathcal{F}_\tau$, then also $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}_\tau$.

1. We have that $\Omega \cap \{\tau \leq k\} = \{\tau \leq k\} \in \mathcal{F}_k$ for all $k \in \{0, 1, \dots, T\}$, so $\Omega \in \mathcal{F}_\tau$.
2. For an $A \in \mathcal{F}_\tau$, we have

$$A^c \cap \{\tau \leq k\} = \{\tau \leq k\} \cap (A^c \cup \{\tau \leq k\}^c) = \{\tau \leq k\} \cap (A \cap \{\tau \leq k\})^c \in \mathcal{F}_k$$

for all $k \in \{0, 1, \dots, T\}$, so $A^c \in \mathcal{F}_\tau$.

3. We show that if $(A_n)_{n\in\mathbb{N}} \in \mathcal{F}_\tau$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}_\tau$. Indeed,

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \cap \{\tau \leq k\} = \bigcup_{n=1}^{\infty} (A_n \cap \{\tau \leq k\}) \in \mathcal{F}_k$$

for all $k \in \{0, 1, \dots, T\}$, so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_\tau$. Hence $\mathcal{F}_\tau$ is a σ-algebra.

(b) For any $A \in \mathcal{F}$, we have that

$$A \cap \{\tau \leq k\} = A \cap \{k_0 \leq k\} = \begin{cases} A & \text{for } k \geq k_0, \\ \emptyset & \text{for } k < k_0. \end{cases}$$

If $A \in \mathcal{F}_\tau$, then all these sets are in $\mathcal{F}_k$, and we get $A \in \mathcal{F}_{k_0}$ for $k = k_0$. Conversely, if $A \in \mathcal{F}_{k_0}$, then all these sets are in $\mathcal{F}_k$ for all $k \in \{0, 1, \dots, T\}$, so that $A \in \mathcal{F}_\tau$.

(c) Call LHS and RHS the left- and right-hand sides of

$$E[Y\,|\,\mathcal{F}_\tau]\,\mathbb{1}_{\{\tau=k\}} = E[Y\,|\,\mathcal{F}_k]\,\mathbb{1}_{\{\tau=k\}},$$

respectively. Now RHS is $\mathcal{F}_k$-measurable since $\{\tau = k\} \in \mathcal{F}_k$. Moreover, $\{\tau = k\} \in \mathcal{F}_\tau$ since for all $\ell \in \{0, 1, \dots, T\}$,

$$\{\tau = k\} \cap \{\tau \le \ell\} = \begin{cases} \{\tau = k\} \in \mathcal{F}_k \subseteq \mathcal{F}_\ell & \text{for } k \le \ell, \\ \emptyset \in \mathcal{F}_\ell & \text{for } k > \ell. \end{cases}$$

So the random variable $\mathbb{1}_{\{\tau=k\}}$ is $\mathcal{F}_\tau$-measurable and

$$\text{LHS} = E\left[Y\mathbb{1}_{\{\tau=k\}}\,\middle|\,\mathcal{F}_\tau\right].$$

Take any $A \in \mathcal{F}_\tau$ and note that $A \cap \{\tau = k\} \in \mathcal{F}_k$. Then,

$$\begin{aligned} E\left[Y\mathbb{1}_{\{\tau=k\}}\mathbb{1}_A\right] &= E\left[Y\mathbb{1}_{A\cap\{\tau=k\}}\right] = E\left[E[Y\,|\,\mathcal{F}_k]\,\mathbb{1}_{A\cap\{\tau=k\}}\right] \\ &= E\left[E[Y\,|\,\mathcal{F}_k]\,\mathbb{1}_{\{\tau=k\}}\mathbb{1}_A\right] = E[\text{RHS}\mathbb{1}_A], \end{aligned}$$

which shows that

$$\text{RHS} = E\left[Y\mathbb{1}_{\{\tau=k\}}\,\middle|\,\mathcal{F}_\tau\right] = \text{LHS } P\text{-a.s.}$$

Notice that in this exercise we did not assume Y to be integrable (or even finite). The integral is nonetheless uniquely defined, but it is good to keep in mind that the (conditional) expectation is only an additive operator on $L^0_+(\mathcal{F})$ – it is not homogeneous, hence it is not linear.

Exercise 7.2 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be an *arbitrage-free* financial market with time horizon T and assume that the bank account process $\widetilde{S}^0 = (\widetilde{S}^0_k)_{k=0,1,\dots,T}$ is given by $\widetilde{S}^0_k = (1+r)^k$ for a constant $r \ge 0$. Denote the set of all EMMs for S^1 by $\mathbb{P}_e(S^1)$. Fix a $\widetilde{K} > 0$. The undiscounted payoff of a *European call option* on $\widetilde{S}^1$ with strike $\widetilde{K}$ and maturity $k \in \{1, \dots, T\}$ is denoted by $\widetilde{C}^E_k$ and given by

$$\widetilde{C}^E_k = \left(\widetilde{S}^1_k - \widetilde{K}\right)^+,$$

whereas the undiscounted payoff of an *Asian call option* on $\widetilde{S}^1$ with strike $\widetilde{K}$ and maturity $k \in \{1, \dots, T\}$ is denoted by $\widetilde{C}^A_k$ and given by

$$\widetilde{C}^A_k := \left(\frac{1}{k}\sum_{j=1}^{k}\widetilde{S}^1_j - \widetilde{K}\right)^+.$$

(a) Fix a $Q \in \mathbb{P}_e(S^1)$ and show that the function $\{1, \dots, T\} \to \mathbb{R}_+$, $k \mapsto E_Q\left[\frac{\widetilde{C}^E_k}{\widetilde{S}^0_k}\right]$ is increasing.

Hint: Use Jensen's inequality for conditional expectations.

(b) Fix a $Q \in \mathbb{P}_e(S^1)$ and show that for all $k = 1, \dots, T$, we have

$$E_Q\left[\frac{\widetilde{C}^A_k}{\widetilde{S}^0_k}\right] \le \frac{1}{k}\sum_{j=1}^{k} E_Q\left[\frac{\widetilde{C}^E_j}{\widetilde{S}^0_j}\right].$$

(c) Fix a $Q \in \mathbb{P}_e(S^1)$ and deduce that for all $k = 1, \dots, T$, we have

$$E_Q\left[\frac{\widetilde{C}^A_k}{\widetilde{S}^0_k}\right] \le E_Q\left[\frac{\widetilde{C}^E_k}{\widetilde{S}^0_k}\right].$$

Interpret this inequality.

Solution 7.2

(a) It clearly suffices to show that for all $k = 1, \ldots, T-1$, we have

$$E_Q\left[\frac{\widetilde{C}^E_{k+1}}{\widetilde{S}^0_{k+1}}\right] \geq E_Q\left[\frac{\widetilde{C}^E_k}{\widetilde{S}^0_k}\right].$$

Fix a $k \in \{1, \ldots, T-1\}$. Using the *tower property* of conditional expectation, *Jensen's inequality* for conditional expectations (for the convex function $x \mapsto x^+$), the fact that S^1 is a Q-martingale and that $r \geq 0$, we get

$$\begin{aligned}
E_Q\left[\frac{\widetilde{C}^E_{k+1}}{\widetilde{S}^0_{k+1}}\right] &= E_Q\left[\left(S^1_{k+1} - \frac{\widetilde{K}}{(1+r)^{k+1}}\right)^+\right] \\
&= E_Q\left[E_Q\left[\left(S^1_{k+1} - \frac{\widetilde{K}}{(1+r)^{k+1}}\right)^+ \,\middle|\, \mathcal{F}_k\right]\right] \\
&\geq E_Q\left[\left(E_Q\left[S^1_{k+1} - \frac{\widetilde{K}}{(1+r)^{k+1}} \,\middle|\, \mathcal{F}_k\right]\right)^+\right] \\
&= E_Q\left[\left(S^1_k - \frac{\widetilde{K}}{(1+r)^{k+1}}\right)^+\right] \\
&\geq E_Q\left[\left(S^1_k - \frac{\widetilde{K}}{(1+r)^k}\right)^+\right] \\
&= E_Q\left[\frac{\widetilde{C}^E_k}{\widetilde{S}^0_k}\right].
\end{aligned}$$

(b) Since the function $x \mapsto x^+$ is convex, we have for $k = 1, \ldots, T$

$$\begin{aligned}
\widetilde{C}^A_k = \left(\frac{1}{k}\sum_{j=1}^k \widetilde{S}^1_j - \widetilde{K}\right)^+ &= \left(\sum_{j=1}^k \frac{1}{k}\left(\widetilde{S}^1_j - \widetilde{K}\right)\right)^+ \\
&\leq \sum_{j=1}^k \frac{1}{k}\left(\widetilde{S}^1_j - \widetilde{K}\right)^+ = \frac{1}{k}\sum_{j=1}^k \widetilde{C}^E_j.
\end{aligned}$$

By *linearity* and *monotonicity* of expectations and since $r \geq 0$, we get

$$\begin{aligned}
E_Q\left[\frac{\widetilde{C}^A_k}{\widetilde{S}^0_k}\right] = E_Q\left[\frac{\widetilde{C}^A_k}{(1+r)^k}\right] &\leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{\widetilde{C}^E_j}{(1+r)^k}\right] \\
&\leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{\widetilde{C}^E_j}{(1+r)^j}\right] = \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{\widetilde{C}^E_j}{\widetilde{S}^0_j}\right].
\end{aligned}$$

(c) Putting the results of (a) and (b) together yields for $k = 1, \ldots, T$

$$E_Q\left[\frac{\widetilde{C}^A_k}{\widetilde{S}^0_k}\right] \leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{\widetilde{C}^E_j}{\widetilde{S}^0_j}\right] \leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{\widetilde{C}^E_k}{\widetilde{S}^0_k}\right] = E_Q\left[\frac{\widetilde{C}^E_k}{\widetilde{S}^0_k}\right].$$

This means nothing else than that for a fixed EMM, the price of an Asian call option on $\widetilde{S}^1$ is smaller than the price of the European call option on the same asset with the same strike price $\widetilde{K}$ and maturity $k \in \{1, \dots, T\}$.

This makes sense also intuitively since the price of a call option is increasing in the volatility of the underlying $\widetilde{A}$ (the probability of ending up in the money, i.e. $P\left[\widetilde{A} > \widetilde{K}\right]$, is higher), and averaging in the Asian call option amounts to reducing volatility of the underlying.

Exercise 7.3 Let $(\widetilde{S}^0, \widetilde{S}^1)$ follow a binomial model with $\widetilde{S}^1_0 = 1$, $u > r > d > -1$ and $T \in \mathbb{N}$. Denote by $(\widehat{S}^0, \widehat{S}^1)$ the market discounted with $\widetilde{S}^1$, i.e.

$$\widehat{S}^0 := \frac{\widetilde{S}^0}{\widetilde{S}^1} \quad \text{and} \quad \widehat{S}^1 := \frac{\widetilde{S}^1}{\widetilde{S}^1} \equiv 1.$$

(a) Show that there exists a unique equivalent martingale measure Q^{**} for $\widehat{S}^0$.

(b) Let Q^* be the unique equivalent martingale measure for $S^1 = \widetilde{S}^1/\widetilde{S}^0$. Show that the density of Q^{**} with respect to Q^* on $\mathcal{F}_T$ is given by

$$\frac{\mathrm{d}Q^{**}}{\mathrm{d}Q^*} = S^1_T.$$

(c) Show that for an *undiscounted* payoff $\widetilde{H} \in L^0_+(\mathcal{F}_T)$, we have

$$\widetilde{S}^0_k \, E_{Q^*}\left[\frac{\widetilde{H}}{\widetilde{S}^0_T} \,\middle|\, \mathcal{F}_k\right] = \widetilde{S}^1_k \, E_{Q^{**}}\left[\frac{\widetilde{H}}{\widetilde{S}^1_T} \,\middle|\, \mathcal{F}_k\right], \quad k = 0, \dots, T.$$

This formula shows that the martingale pricing method is invariant under a so-called *change of numéraire*.
Hint: Use Bayes' formula (Lemma II.3.1) in the lecture notes.

Solution 7.3 Assume without loss of generality that $\Omega = \{1,2\}^T$ and $Y_k((x_1, \dots, x_T)) = 1 + y_{x_k}$, where $y_1 := d$ and $y_2 := u$.

(a) **First method.**

Any measure $Q \approx P$ on $\mathcal{F}_T$ can be described by its transition probabilities q_{x_1}, q_{x_1,x_2}, …, $q_{x_1,\dots,x_T}$, where x_1, …, $x_k \in \{1,2\}$ and

$$\begin{aligned} q_{x_1} &:= Q[Y_1 = 1 + y_{x_1}], \\ q_{x_1,\dots,x_k} &:= Q[Y_k = 1 + y_{x_k} \,|\, Y_1 = 1 + y_{x_1}, \dots, Y_{k-1} = 1 + y_{x_{k-1}}], \quad k = 2, \dots, T. \end{aligned}$$

Since $\widehat{S}^0$ only attains a finite number of values and hence is bounded and since it is also adapted, it is a Q-martingale if and only if for all $k = 0, \dots, T-1$, we have

$$E_Q\left[\widehat{S}^0_{k+1} \,\middle|\, \mathcal{F}_k\right] = \widehat{S}^0_k \quad Q\text{-a.s.} \tag{1}$$

By the definition of $\widehat{S}^0$ and since it is strictly positive, the martingale property (1) holds if and only if

$$E_Q\left[\widehat{S}^0_k \frac{1+r}{Y_{k+1}} \,\middle|\, \mathcal{F}_k\right] = \widehat{S}^0_k \quad Q\text{-a.s.},$$

which holds if and only if

$$E_Q\left[\frac{1+r}{Y_{k+1}} \,\middle|\, \mathcal{F}_k\right] = 1 \quad Q\text{-a.s.}$$

Note that we do not know a priori whether the Y_k are independent under Q. Since $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_k = \sigma(Y_1, \ldots, Y_k)$ for $k = 1, \ldots T$, and Y_k only takes two values, $\widehat{S}^0$ is an Q-martingale if and only if $E_Q\left[\frac{1+r}{Y_1}\right] = 1$ and for all $k \in \{1, \ldots, T-1\}$ and $x_1, \ldots, x_k \in \{1, 2\}$ we have

$$E_Q\left[\frac{1+r}{Y_{k+1}}\middle| Y_1 = 1 + y_{x_1}, \ldots, Y_k = 1 + y_{x_k}\right] = 1.$$

Now, we have

$$\begin{aligned} E_Q\left[\frac{1+r}{Y_1}\right] = 1 \quad &\iff \quad \frac{1+r}{1+d}q_1 + \frac{1+r}{1+u}(1-q_1) = 1 \\ &\iff \quad \left(\frac{1+u}{1+d} - 1\right) q_1 = \frac{1+u}{1+r} - 1 \\ &\iff \quad \frac{u-d}{1+d} q_1 = \frac{u-r}{1+r} \\ &\iff \quad q_1 = \frac{1+d}{1+r}\frac{u-r}{u-d}. \end{aligned}$$

Similarly, for all $k \in \{1, \ldots, T-1\}$ and all x_1, …, $x_k \in \{1, 2\}$, we have

$$E_Q\left[\frac{1+r}{Y_{k+1}}\middle| Y_1 = 1 + y_{x_1}, \ldots, Y_k = 1 + y_{x_k}\right] = 1 \iff \quad q_{x_1,\ldots,x_k,1} = \frac{1+d}{1+r}\frac{u-r}{u-d}.$$

Note that $q_{x_1,\ldots,x_k,1}$ does not depend on x_1, …, x_k and equals q_1. Hence, we may conclude that there exists a unique equivalent martingale measure Q^{**} for $\widehat{S}^0$, under which $Y_1, \ldots, Y_k$ are i.i.d. and we have

$$Q^{**}[Y = 1+d] = \frac{1+d}{1+r}\frac{u-r}{u-d} =: q_1^{**} \quad \text{and} \quad Q^{**}[Y = 1+u] = \frac{1+u}{1+r}\frac{r-d}{u-d} =: q_2^{**}.$$

Second method:

For $k = 1$, …, T, set $\widehat{Y}_k = \frac{1+r}{Y_k}$. Then $\widehat{S}^1_k = 1^k = 1$ and $\widehat{S}^0_k = \prod_{j=1}^k \widehat{Y}_j$ for $k = 0$, …, T, where the $\widehat{Y}_k$ are independent under P and take the two values $1 + \hat{u}$ and $1 + \hat{d}$ with probability p_1 and p_2, respectively, where $\hat{u} = \frac{r-d}{1+d} > 0$ and $\hat{d} = \frac{r-u}{1+u} < 0$. In conclusion, $(\widehat{S}^1, \widehat{S}^0)$ can be viewed as a binomial model with $\hat{u} > \hat{r} = 0 > \hat{d}$.

Variant (i):

Recalling that $\hat{u}$ corresponds to p_1 and $\hat{d}$ to p_2, it follows from Corollary 2.1.4 in the lecture notes that the unique equivalent martingale measure Q^{**} for $\widehat{S}^1$ is given by

$$Q^{**}\left[\{(x_1, \ldots, x_T\}\right] := \prod_{k=1}^T q^{**}_{x_k}, \quad x_1, \ldots, x_T \in \{1, 2\},$$

where

$$\begin{aligned} q_1^{**} &= \frac{\hat{r} - \hat{d}}{\hat{u} - \hat{d}} = \frac{\frac{u-r}{1+u}}{\frac{r-d}{1+d} - \frac{r-u}{1+u}} = \frac{1+d}{1+r}\frac{u-r}{u-d} \\ q_2^{**} &= \frac{\hat{u} - \hat{r}}{\hat{u} - \hat{d}} = \frac{\frac{r-d}{1+d}}{\frac{r-d}{1+d} - \frac{r-u}{1+u}} = \frac{1+u}{1+r}\frac{r-d}{u-d}. \end{aligned}$$

Variant (ii):

Since we have a binomial model, we know there can be at most one EMM. So we are looking for the unique strictly positive Q^*-martingale $Z^{Q^{**};Q^*}$ starting at 1 such that

$$\frac{\widetilde{S}^0}{\widetilde{S}^1} Z^{Q^{**};Q^*} \text{ is a } Q^*\text{-martingale.}$$

But we already know that $\frac{\widetilde{S}^1}{\widetilde{S}^0}$ is a Q^*-martingale and strictly positive. Hence by uniqueness,

$$Z^{Q^{**};Q^*} = \frac{\widetilde{S}^1}{\widetilde{S}^0} = S^1.$$

(b) As explained in part (a), variant (ii), the density process $Z^{Q^{**};Q^*}$ of the unique EMM Q^{**} with respect to Q^* is given by

$$Z_k^{Q^{**};Q^*} = S_k^1, \quad \forall\ k \in \{0, \ldots, T\}.$$

By Lemma II.3.1, this give us that for all $A \in \mathcal{F}_T$

$$Q^{**}[A] = E_{Q^*}[Z_T^{Q^{**};Q^*} \mathbb{1}_A] = E_{Q^*}[S_T^1 \mathbb{1}_A]$$

and thus that on $\mathcal{F}_T$,

$$\frac{\mathrm{d}Q^{**}}{\mathrm{d}Q^*} = S_T^1.$$

(c) In order to simplify the notation, denote by Z the density process $Z^{Q^{**};Q^*}$ of Q^{**} with respect to Q^*. Let now $k \in \{1, \ldots, T\}$ and $\widetilde{H} \in L_+^0(\mathcal{F}_T)$ and recall that

$$Z = S^1 = \frac{\widetilde{S}^1}{\widetilde{S}^0} \quad Q^*\text{-a.s.}$$

Thus, by the Bayes formula we get for $k = 0, \ldots, T$

$$\widetilde{S}_k^1\, E_{Q^{**}}\left[\frac{\widetilde{H}}{\widetilde{S}_T^1} \,\middle|\, \mathcal{F}_k\right] = \frac{\widetilde{S}_k^1}{Z_k} E_{Q^*}\left[\frac{Z_T \widetilde{H}}{\widetilde{S}_T^1} \,\middle|\, \mathcal{F}_k\right] = \widetilde{S}_k^0\, E_{Q^*}\left[\frac{\widetilde{H}}{\widetilde{S}_T^0} \,\middle|\, \mathcal{F}_k\right].$$