



Mathematical Foundations for Finance

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Exercise 8

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# Normal Distribution

Due to the definition of Brownian motion, normal distribution plays an important role in the computations that involve it.

The density of  $\mathcal{N}(0, 1)$  will be denoted by  $\phi$  and takes the following form:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

The cumulative distribution function (cdf) of  $\mathcal{N}(0, 1)$  will be denoted  $\Phi$  and cannot be expressed in a closed form.

We can define  $\mathcal{N}(\mu, \sigma^2)$  as the distribution of  $X = \sigma Z + \mu$  for a  $\mu \in \mathbb{R}$  and  $\sigma > 0$ . This means that for  $F_X(x)$  and  $f_X(x)$ , denoting the CDF and the density of  $X$  respectively, we have that

$$F_X(x) = P[\sigma Z + \mu \leq x] = P\left[Z \leq \frac{x - \mu}{\sigma}\right] = \Phi\left(\frac{x - \mu}{\sigma}\right),$$
$$f_X(x) = \frac{d}{dx} \Phi\left(\frac{x - \mu}{\sigma}\right) = \frac{1}{\sigma} \phi\left(\frac{x - \mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right).$$

# Normal Distribution

Let  $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$  and  $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$  be *independent* random variables (defined on the same probability space). Then

1.  $E[|X|^k] < \infty$  for all  $k \in \mathbb{N}$

This can be easily seen since the exponential function grows faster than any power.

2.  $E[(X - \mu_1)^k] = 0$  for all odd  $k \in \mathbb{N}$

This follows from the symmetry of  $\phi$  – the function  $x^k \phi\left(\frac{x}{\sigma}\right)$  is odd.

3.  $aX + b \sim \mathcal{N}(a\mu_1 + b, \sigma_1^2 a^2)$  for any  $a, b \in \mathbb{R}$

Affine transformations of normal random variables retain normality.

4.  $X \pm Y \sim \mathcal{N}(\mu_1 \pm \mu_2, \sigma_1^2 + \sigma_2^2)$

Sums of *independent* normal random variables retain normality.

# Normal Distribution

The last two properties can be easily shown using the so-called *moment generating function* (MGF), defined for a random variable  $X$  by

$$M_X(t) = E [\exp(tX)], \quad t \in \mathbb{R},$$

whenever this expectation exists.

The MGF of  $X \sim \mathcal{N}(\mu, \sigma^2)$  is

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

and its derivation demonstrates the technique of completing the square, which is frequently useful when it comes to normal distribution.

Additionally, we will often work with random variables of the form  $Y = \exp(X)$  with  $X \sim \mathcal{N}(\mu, \sigma^2)$  (i.e.  $Y$  is a lognormal r.v.), so the knowledge of the MGF of normal distribution comes handy when computing  $E[Y]$ , as we have that

$$E[Y] = M_X(1).$$

## Theorem 1

For any local martingale  $M = (M_t)_{t \geq 0}$  null at 0, there exists a unique adapted increasing RCLL process  $[M] = ([M]_t)_{t \geq 0}$  null at zero with  $\Delta[M] = (\Delta M)^2$  having the property that  $M^2 - [M]$  is a local martingale.

- We call this process  $[M]$  the *square bracket process* or *optional quadratic variation*.
- **Important:**  $[M]$  is not a unique process such that  $M^2 - [M]$  is a local martingale. It is just unique among all processes satisfying the conditions in the above theorem.
- This process and its properties are of significant importance because of its importance in stochastic analysis – it comes up in the famous Itô's formula for instance.

## Definition 2 (Covariation process)

Let  $M$  and  $N$  be two local martingales that are both null at 0. We define the *covariation process* of  $M$  and  $N$  by

$$[M, N] = \frac{1}{4} ([M + N] - [M - N]).$$

- $[M, N]$  is the unique adapted RCLL process that is null at 0, of finite variation and with  $\Delta[M, N] = \Delta M \Delta N$ .
- Note that  $[M]$  and  $[N]$  are increasing and therefore of finite variation.  $[M, N]$  is not necessarily increasing anymore, but still of finite variation like  $[M]$  and  $[N]$ . Compare this with variance and covariance.
- Similarly to the optional quadratic variation, covariation is of significant importance in stochastic analysis. Naturally, it occurs when we are dealing with multivariate stochastic processes.

# Sharp Bracket Process

If the local martingale  $M$  is locally square-integrable, then  $[M]$  is locally integrable and it admits a unique increasing predictable process  $\langle M \rangle = (\langle M \rangle)_{t \geq 0}$  null at zero such that  $[M] - \langle M \rangle$  is a local martingale.

- Since sums of local martingales (w.r.t. the same probability measure and filtration) are again local martingales, we also have that

$$M^2 - \langle M \rangle = (M^2 - [M]) + ([M] - \langle M \rangle)$$

is a local martingale.

- The process  $\langle M \rangle$  is called the *predictable compensator* or the *sharp bracket process* of  $M$ .
- As before,  $\langle M \rangle$  is only unique among the processes with the aforementioned properties. In particular we do not have that  $\Delta \langle M \rangle = (\Delta M)^2$ .

Thank you for your attention!