Mathematical Foundations for Finance

Exercise sheet 8

Exercise 8.1 Let $W = (W_t)_{t \geq 0}$ be a Brownian motion (BM) defined on some probability space $(\Omega, \mathcal{F}, P)$ (without filtration). Show that

(a) $W^1 := -W$ is a BM.

(b) $W_t^2 := W_{T+t} - W_T$, $t \geq 0$, is a BM for any $T \in (0, \infty)$.

(c) $W^3 := \alpha B + \sqrt{1-\alpha^2} B'$ is a BM, where B and B' are two independent BMs and $\alpha \in [0,1]$.

(d) Show that the independence of B and B' in (c) cannot be omitted, i.e., if B and B' are *not* independent, then W^3 need not be a BM. Give two examples.

Solution 8.1 We first recall the definition of a Brownian motion (without filtration) in order to know what needs to be checked. A *Brownian motion* with respect to P is a real-valued stochastic process $W = (W_t)_{t \geq 0}$ such that

(BM0) $W_0 = 0$ P-a.s.

(BM1′) For any $n \in \mathbb{N}$ and any times $0 = t_0 < t_1 < \cdots < t_n < \infty$, the increments $W_{t_i} - W_{t_{i-1}}$ are independent and normally distributed with variance $t_i - t_{i-1}$ under P, i.e.

$$W_{t_i} - W_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1}) \text{ for } i = 1, \dots, n.$$

(BM2) W has P-a.s. continuous trajectories.

(a) We check (BM0), (BM1′) and (BM2) separately.

(BM0) This is clear since $W_0^1 = -W_0 = 0$ P-a.s.

(BM1′) Let $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$. Then we have

$$W_{t_i}^1 - W_{t_{i-1}}^1 = -(W_{t_i} - W_{t_{i-1}}), \quad i = 1, \dots, n,$$

which are independent under P. Since $X \sim \mathcal{N}(0, \sigma^2)$ if and only if $-X \sim \mathcal{N}(0, \sigma^2)$, we also conclude that $W_{t_i}^1 - W_{t_{i-1}}^1 \sim \mathcal{N}(0, t_i - t_{i-1})$.

(BM2) This is trivial, since $W^1 = -W$. The sign does not alter continuity.

(b) We check (BM0), (BM1′) and (BM2) separately.

(BM0) We obviously have $W_0^2 = W_T - W_T = 0$ P-a.s.

(BM1′) Let $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$. Then we have

$$\begin{aligned} W_{t_i}^2 - W_{t_{i-1}}^2 &= W_{T+t_i} - W_T - (W_{T+t_{i-1}} - W_T) \\ &= W_{T+t_i} - W_{T+t_{i-1}}, \quad i = 1, \dots, n. \end{aligned}$$

Denoting $t_i' = T + t_i$, we see from the definition (BM1′) that the increments of W^2 are independent under P, and since $t_i' - t_{i-1}' = t_i - t_{i-1}$, we also conclude that

$$W_{t_i}^2 - W_{t_{i-1}}^2 \sim \mathcal{N}(0, t_i - t_{i-1}), \text{ for } i = 1, \dots, n.$$

(BM2) This is again easy, since W^2 is simply W shifted in time by T minus a random variable which does not depend on t.

(c) We check (BM0), (BM1′) and (BM2) separately.

(BM0) $W^3_0 = \alpha B_0 + \sqrt{1-\alpha^2} B'_0 = 0$ P-a.s., since both B_0 and B'_0 are equal to 0 P-a.s.

(BM1′) Let $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$. Then we have

$$W^3_{t_i} - W^3_{t_{i-1}} = \alpha \left(B_{t_i} - B_{t_{i-1}} \right) + \sqrt{1-\alpha^2} \left(B'_{t_i} - B'_{t_{i-1}} \right), \quad i = 1, \ldots, n.$$

Since B and B' are independent under P, we conclude that the right-hand side is an independent family of random variables. Since B and B' are BMs, we additionally have that

$$B_{t_i} - B_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1}), \quad i = 1, \ldots, n,$$
$$B'_{t_i} - B'_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1}), \quad i = 1, \ldots, n.$$

Recall the general fact that if $X \sim \mathcal{N}(0, \sigma^2)$ and $Y \sim \mathcal{N}(0, \eta^2)$ are independent, then we have for any linear combination $s_1 X + s_2 Y$ that

$$s_1 X + s_2 Y \sim \mathcal{N}(0, s_1^2 \sigma^2 + s_2^2 \eta^2).$$

Using this, we conclude that

$$\begin{aligned}\alpha \left(B_{t_i} - B_{t_{i-1}} \right) &+ \sqrt{1-\alpha^2} \left(B'_{t_i} - B'_{t_{i-1}} \right) \\ &\sim \mathcal{N}(0, \alpha^2 (t_i - t_{i-1}) + (1-\alpha^2)(t_i - t_{i-1})) \\ &= \mathcal{N}(0, t_i - t_{i-1}).\end{aligned}$$

(BM2) This is evident, since W^3 is a linear combination of two processes whose paths are P-a.s. continuous.

(d) Two possible choices are $B = \pm B'$. In this case we have

$$W^3 = (\alpha \pm \sqrt{1-\alpha^2}) B,$$

which is not a Brownian motion because $W^3_1 \sim \mathcal{N}(0, \left((\alpha \pm \sqrt{1-\alpha^2}) \right)^2)$ and $(\alpha \pm \sqrt{1-\alpha^2}))^2 \neq 1$ in general.

Exercise 8.2 Let $(\Pi_n)_{n \in \mathbb{N}}$ be a sequence of refining partitions of $[a, b] \subseteq \mathbb{R}$ (in the sense that $\Pi_n \subseteq \Pi_{n+1}$ for all $n \in \mathbb{N}$) with $|\Pi_n| \to 0$ as $n \to \infty$. Let $p > 0$. We define for a function $f : \mathbb{R} \to \mathbb{R}$ its p-variation on $[a, b]$ along the sequence $(\Pi_n)_{n \in \mathbb{N}}$ as

$$V_p^{(a,b)}(f) := \lim_{n \to \infty} \sum_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})|^p,$$

assuming that the limit exists. Assume additionally that f is continuous on $[a, b]$.

(a) Show that if $V_{p^*}^{(a,b)}(f)$ is finite and non-zero for some $p^* > 0$, then $V_p^{(a,b)}(f) = \infty$ for all $p < p^*$. *Hint: Make sure to use the continuity of f. Use also that every function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on a closed and bounded interval $[a, b]$ is also uniformly continuous on $[a, b]$.*

(b) Show that if $V_{p^*}^{(a,b)}(f)$ is finite and non-zero for some $p^* > 0$, then $V_p^{(a,b)}(f) = 0$ for all $p > p^*$.

Solution 8.2

(a) We show this by contradiction. Suppose that $V_p^{(a,b)}(f) < K$ for some $p < p^*$ and $K \in \mathbb{R}_+$. We then have that

$$\begin{aligned}
V_{p^*}^{(a,b)}(f) &= \lim_{n\to\infty} \sum_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})|^{p^*} \\
&\leq \lim_{n\to\infty} \left[\left(\sup_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})| \right)^{p^*-p} \sum_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})|^p \right] \\
&= \lim_{n\to\infty} \left(\sup_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})| \right)^{p^*-p} \lim_{n\to\infty} \sum_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})|^p \\
&= \lim_{n\to\infty} \left(\sup_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})| \right)^{p^*-p} V_p^{(a,b)}(f) \\
&\leq K \lim_{n\to\infty} \left(\sup_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})| \right)^{p^*-p} \\
&= K \left(\lim_{n\to\infty} \sup_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})| \right)^{p^*-p} \\
&= 0,
\end{aligned}$$

where the second to last equality holds since the function $x \mapsto x^k$ is continuous, and the last equality holds because every function that is continuous on a compact interval is also uniformly continuous on this interval, which means that

$$\lim_{n\to\infty} \sup_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})| = 0.$$

Since we have that $V_p^{(a,b)}(f) \geq 0$ for any $p > 0$ by definition and we assumed that the p^*-variation of f is finite and non-zero, this gives a contradiction. The p-variation of f must therefore be infinite for all $p < p^*$.

(b) Let us assume without loss of generality that $V_{p^*}^{(a,b)}(f) = K$ for some $K \in \mathbb{R}_+$. Using exactly the same reasoning as in (a), we have for $p > p^*$ that

$$\begin{aligned}
V_p^{(a,b)}(f) &= \lim_{n\to\infty} \sum_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})|^p \\
&\leq \lim_{n\to\infty} \left(\sup_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})| \right)^{p-p^*} \sum_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})|^{p^*} \\
&= \lim_{n\to\infty} \left(\sup_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})| \right)^{p-p^*} \lim_{n\to\infty} \sum_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})|^{p^*} \\
&= \left(\lim_{n\to\infty} \sup_{t_i \in \Pi_n} |f(t_i) - f(t_{i-1})| \right)^{p-p^*} V_{p^*}^{(a,b)}(f) \\
&= 0,
\end{aligned}$$

which is what we wanted to show.

Exercise 8.3 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some sufficiently rich filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous convex function. Show that if the stochastic process $\big(f(W_t)\big)_{t\geq 0}$ is integrable, then it is a $(P, \mathbb{F})$-submartingale.
Hint: We have done something similar in discrete time.

(b) Given a $(P, \mathbb{F})$-martingale $(M_t)_{t\geq 0}$ and a measurable function $g : \mathbb{R}_+ \to \mathbb{R}$, show that the process

$$\big(M_t + g(t)\big)_{t\geq 0}$$

is a $(P, \mathbb{F})$-supermartingale if and only if g is decreasing, and a $(P, \mathbb{F})$-submartingale if and only if g is increasing.

(c) Show that the following stochastic processes are $(P, \mathbb{F})$-submartingales but not martingales:

(i) W^2,

(ii) $e^{\alpha W}$ for any $\alpha \in \mathbb{R}$.

Hint: Use the result from (a) and (b), respectively.

(d) Show that any $(P, \mathbb{F})$-local martingale which is null at 0 and uniformly bounded from below is a $(P, \mathbb{F})$-supermartingale.
Hint: We have done this in discrete time already.

Solution 8.3

(a) First recall that W is a $(P, \mathbb{F})$-martingale. Adaptedness is clear since f is assumed to be continuous. Integrability is assumed as well. Then by Jensen's inequality for conditional expectations, we can compute

$$E\left[f(W_t)\,|\,\mathcal{F}_s\right] \geq f\big(E\left[W_t\,|\,\mathcal{F}_s\right]\big) = f(W_s) \qquad P\text{-a.s.}$$

for all $t \geq s$, and thus conclude that $f(W)$ is a $(P, \mathbb{F})$-submartingale.

(b) For any measurable function $g : \mathbb{R}_+ \to \mathbb{R}$, we have that $M_t + g(t)$ is $\mathcal{F}_t$-measurable and

$$E\left[|M_t + g(t)|\right] \leq E\left[|M_t|\right] + E\left[|g(t)|\right] = E\left[|M_t|\right] + |g(t)| < \infty.$$

Hence $\big(M_t + g(t)\big)_{t\geq 0}$ is adapted and integrable. We can then compute

$$E\left[M_t + g(t)\,|\,\mathcal{F}_s\right] = E\left[M_t\,|\,\mathcal{F}_s\right] + g(t) = M_s + g(s) + g(t) - g(s) \qquad P\text{-a.s.}$$

for all $t \geq s$. As a result, $\big(M_t + g(t)\big)_{t\geq 0}$ has the $(P, \mathbb{F})$-supermartingale property, i.e.

$$E\left[M_t + g(t)\,|\,\mathcal{F}_s\right] \leq M_s + g(s) \qquad P\text{-a.s.}$$

for all $t > s$, if and only if g is decreasing. Analogously, $\big(M_t + g(t)\big)_{t\geq 0}$ has the $(P, \mathbb{F})$-submartingale property, i.e.

$$E\left[M_t + g(t)\,|\,\mathcal{F}_s\right] \geq M_s + g(s) \qquad P\text{-a.s.}$$

for all $t > s$, if and only if g is increasing.

(c) (i) Note that $W_t^2 = W_t^2 - t + g(t)$, where $g(t) := t$. By Proposition IV.2.2. in the lecture notes, we know that $\big(W_t^2 - t\big)_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale; hence, using that g is increasing, by (b) we can conclude that W^2 is a $(P, \mathbb{F})$-submartingale.

In order to show that W^2 is not a martingale, we can use the martingale property of $\left(W_t^2 - t\right)_{t\geq 0}$ to compute

$$E\left[W_t^2 \,\middle|\, \mathcal{F}_s\right] = E\left[W_t^2 - t \,\middle|\, \mathcal{F}_s\right] + t = W_s^2 - s + t > W_s^2 \qquad P\text{-a.s.},$$

showing that W^2 is not a $(P, \mathbb{F})$-martingale.

Alternatively, by Jensen's inequality for conditional expectations we have that

$$E\left[W_t^2 \,\middle|\, \mathcal{F}_s\right] \geq \left(E\left[W_t \,\middle|\, \mathcal{F}_s\right]\right)^2 = W_s^2,$$

and the inequality is strict with positive probability because $x \mapsto x^2$ is strictly convex and W_t is not P-a.s. constant. So W^2 is a submartingale but not a martingale. The same argument can be used for (c) with $x \mapsto e^{\alpha x}$.

(ii) Adaptedness is clear since the transformation $x \mapsto e^{\alpha x}$ is continuous, and since we know that $W_t \overset{d}{=} W_t - W_0$ is $\mathcal{N}(0,t)$-distributed, the random variable $e^{\alpha W_t}$ is integrable. Noting that $x \mapsto e^{\alpha x}$ is also a convex function, we can then apply (a) to conclude that $e^{\alpha W}$ is a $(P, \mathbb{F})$-submartingale.

Next, Proposition IV.2.2. in the lecture notes gives us that $\left(e^{\alpha W_t - \frac{1}{2}\alpha^2 t}\right)_{t\geq 0}$ is a $(P, \mathbb{F})$-martingale; hence, we can compute

$$E\left[e^{\alpha W_t} \,\middle|\, \mathcal{F}_s\right] = E\left[e^{\alpha W_t - \frac{1}{2}\alpha^2 t} \,\middle|\, \mathcal{F}_s\right] e^{\frac{1}{2}\alpha^2 t} = e^{\alpha W_s} e^{\frac{1}{2}\alpha^2 (t-s)} > e^{\alpha W_s} \qquad P\text{-a.s.},$$

showing that $e^{\alpha W}$ is not a $(P, \mathbb{F})$-martingale.

(d) Let $(X_t)_{t\geq 0}$ be a $(P, \mathbb{F})$-local martingale null at 0 and uniformly bounded from below by $-a \leq 0$ and denote by $(\tau_n)_{n\in\mathbb{N}}$ a localizing sequence. Since $\lim_{n\to\infty} \tau_n = \infty$ P-a.s., we have

$$\lim_{n\to\infty} X_{t\wedge\tau_n} = X_t \qquad P\text{-a.s.}$$

Moreover, since $(X_t)_{t\geq 0}$ is uniformly bounded from below by $-a$, we have that $X_{t\wedge\tau_n} \geq -a$ and thus $0 \leq |X_{t\wedge\tau_n}| \leq X_{t\wedge\tau_n} + 2a$ for all $n \in \mathbb{N}$. By Fatou's lemma, we can then compute

$$E\left[|X_t|\right] = E\left[\lim_{n\to\infty} |X_{t\wedge\tau_n}|\right] \leq \liminf_{n\to\infty} E\left[|X_{t\wedge\tau_n}|\right] \leq \liminf_{n\to\infty} E\left[X_{t\wedge\tau_n}\right] + 2a = 2a < \infty,$$

where the last equality uses the martingale property of X^{τ_n} and the fact that it is null at 0. We have thus proved integrability. Since adaptedness is clear by the definition of a local martingale, it only remains to show the $(P, \mathbb{F})$-supermartingale property. Using again that $X_{t\wedge\tau_n} \geq -a$ for all $n \in \mathbb{N}$, we can apply Fatou's lemma to obtain for $t > s$

$$E\left[X_t \,\middle|\, \mathcal{F}_s\right] = E\left[\lim_{n\to\infty} X_{t\wedge\tau_n} \,\middle|\, \mathcal{F}_s\right] \leq \liminf_{n\to\infty} E\left[X_{t\wedge\tau_n} \,\middle|\, \mathcal{F}_s\right] = X_s,$$

as desired.