Mathematical Foundations for Finance

Exercise sheet 9

Please hand in your solutions until Tuesday, 20/11/2018, 18:00 into your assistant's box next to HG G 53.2.

Exercise 9.1 Let $(Y_k)_{k\in\mathbb{N}}$ be a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and consider the filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\mathbb{N}_0}$ given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(Y_1, \dots, Y_k)$ for all $k \in \mathbb{N}$. Let $E[Y_k] = \mu$ and $\operatorname{Var}(Y_k) = \sigma^2$ for all $k \in \mathbb{N}$ with $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Define additionally $X = (X_k)_{k\in\mathbb{N}_0}$ by

$$X_k = \sum_{j=1}^{k} Y_j \quad \text{for all } k \in \mathbb{N}_0,$$

and assume that X is adapted to $\mathbb{F}$.

(a) Show that for any $\mathbb{F}$-adapted integrable process $Z = (Z_k)_{k\in\mathbb{N}_0}$, there exists a P-a.s. unique decomposition of Z into $Z = M + A$ with $M = (M_k)_{k\in\mathbb{N}_0}$ a $(P, \mathbb{F})$-martingale and $A = (A_k)_{k\in\mathbb{N}_0}$ an $\mathbb{F}$-predictable integrable process with $A_0 = 0$.
Hint: This is the Doob decomposition. Show the existence by construction.

(b) Using (a), explicitly derive the processes M and A in the Doob decomposition of X.

(c) Explicitly derive the optional quadratic variation $[M] = \big([M]_k\big)_{k\in\mathbb{N}_0}$ of the square-integrable martingale M from (b), and show that $M^2 - [M]$ is a martingale.
Hint: See Theorem V.1.1 in the lecture notes, and use that due to the condition $\Delta[M] = (\Delta M)^2$, we must have that $[M]_k - [M]_{k-1} = (M_k - M_{k-1})^2$.

(d) Explicitly derive the predictable compensator $\langle M\rangle = \big(\langle M\rangle_k\big)_{k\in\mathbb{N}_0}$ of the process M from (b).
Hint: See the remark on page 79 in the lecture notes. Also use that if M is a square-integrable martingale, then $\langle M\rangle$ is integrable.

Exercise 9.2 A *Poisson process* with parameter $\lambda > 0$ with respect to a probability measure P and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a (real-valued) stochastic process $N = (N_t)_{t\geq 0}$ which is adapted to $\mathbb{F}$, $N_0 = 0$ P-a.s. and satisfies the following two properties:

(PP1) For $0 \leq s < t$, the *increment* $N_t - N_s$ is independent (under P) of $\mathcal{F}_s$ and is (under P) *Poisson-distributed* with parameter $\lambda(t-s)$, i.e.

$$P[N_t - N_s = k] = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}, \quad k \in \mathbb{N}_0.$$

(PP2) N is a *counting process* with jumps of size 1, i.e. for P-almost all $\omega \in \Omega$, the function $t \mapsto N_t(\omega)$ is right-continuous with left limits (RCLL), piecewise constant, $\mathbb{N}_0$-valued, and increases by jumps of size 1.

Poisson processes form the cornerstone of *jump processes*, which are of importance in advanced financial modeling. Show that the following processes are $(P, \mathbb{F})$-martingales:

(a) $\widetilde{N}_t := N_t - \lambda t$, $t \geq 0$. This process is also called a *compensated Poisson process.*
Hint: If $X \sim Poi(\lambda)$, then $E[X] = \lambda$.

(b) $\widetilde{N}_t^2 - N_t$, $t \geq 0$, and $\widetilde{N}_t^2 - \lambda t$, $t \geq 0$. Use these results to derive $[\widetilde{N}]$ and $\langle \widetilde{N} \rangle$.
Hint: If $X \sim Poi(\lambda)$, then $\mathrm{Var}(X) = \lambda$.

(c) $S_t := e^{N_t \log(1+\sigma) - \lambda \sigma t}$, $t \geq 0$, where $\sigma > -1$. S is also called a *geometric Poisson process.*

Exercise 9.3 Let $W = (W_t)_{t \geq 0}$ be a Brownian motion. Let $a, b > 0$ and define

$$\tau_a = \inf\{t \geq 0 \,|\, W_t > a\},$$
$$\sigma_{a,b} = \inf\{t \geq 0 \,|\, W_t > a + bt\}.$$

(a) Show that for $\tau \in \{\tau_a, \sigma_{a,b}\}$ and all $\alpha \in \mathbb{R}$ we have that

$$E\left[e^{\alpha W_{\tau \wedge t} - \frac{1}{2}\alpha^2 (\tau \wedge t)}\right] = 1.$$

(b) Using your result from (a) show that

$$e^{\alpha a} E\left[e^{-\frac{1}{2}\alpha^2 \tau_a}\right] = 1,$$

and use this to conclude by an appropriate choice of α that the Laplace transform ϕ_{τ_a} of τ_a is given by

$$\phi_{\tau_a}(\lambda) := E\left[e^{-\lambda \tau_a}\right] = e^{-a\sqrt{2\lambda}}, \quad \lambda > 0.$$

Hint 1: Make use of dominated convergence theorem.
Hint 2: Use that $W_{\tau_a} = a$ P-a.s.; we will show this in another exercise sheet.

(c) Using your result from (a) show that

$$e^{\alpha a} E\left[e^{(ab - \frac{1}{2}\alpha^2)\sigma_{a,b}}\right] = 1,$$

and use this to conclude by an appropriate choice of α that the Laplace transform $\phi_{\sigma_{a,b}}$ of $\sigma_{a,b}$ is given by

$$\phi_{\sigma_{a,b}}(\lambda) := E\left[e^{-\lambda \sigma_{a,b}}\right] = e^{-a(b + \sqrt{b^2 + 2\lambda})}, \quad \lambda > 0.$$

Hint 1: Make use of dominated convergence theorem.
Hint 2: Use that $W_{\sigma_{a,b}} = a + b\sigma_{a,b}$ P-a.s.

(d) Show that τ_a is P-a.s. finite for any $a > 0$ and that $\sigma_{a,b}$ takes the value of $+\infty$ with a positive probability for any $a, b > 0$.