ETH zürich

Mathematical Foundations for Finance

Exercise 9

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Stochastic Integration Roadmap

The following steps briefly describe the way we get the definition of the stochastic integral $(H \cdot M)$ for a large class of integrands H and integrators M:

1. We define $(H \cdot M)$ for an $H \in b\mathcal{E}$ a bounded elementary process and a square-integrable martingale $M \in \mathcal{H}^2$.

2. We extend the definition to $H \in L^2(M)$, the closure of $b\mathcal{E}$. This works well for $M \in \mathcal{M}_0^2$, a subset of the set of all square-integrable martingales.

3. We extend the definition to $H \in L^2_{\text{loc}}(M)$ and $M \in \mathcal{M}^2_{0,\text{loc}}$ via localization.

4. We extend this definition to $H \in L^2_{\text{loc}}(M)$ and an arbitrary local martingale M.

5. We utilize the previous and define $(H \cdot X)$ for H predictable and locally bounded and for X a general semimartingale.

We look at all of these in more detail later.

Bounded Elementary Processes

Definition 1 (Bounded elementary process)

A *bounded elementary process* on $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a stochastic process $H = (H_t)_{t\geq 0}$ of the form

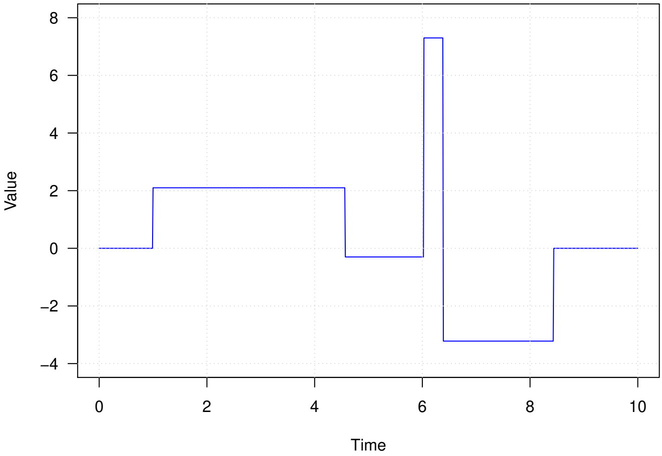
$$H_t(\omega) = \sum_{i=0}^{n-1} h_i(\omega)\mathbb{1}_{(t_i,t_{i+1}]}(t)$$

for some $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \ldots < t_n < \infty$ and h_i a bounded $\mathcal{F}_{t_i}$-measurable random variable for all $i \in \{0, 1, \ldots, n-1\}$.

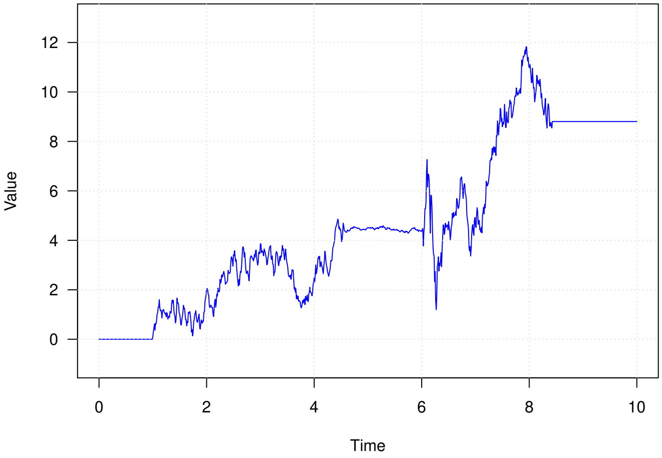
- The set of all bounded elementary processes is denoted by $b\mathcal{E}$.
- We can define $(H \cdot M) = \big((H \cdot M)_t\big)_{t\geq 0}$ for $H \in b\mathcal{E}$ and $M = (M_t)_{t\geq 0}$ a square-integrable martingale by

$$\big(H \cdot M\big)_t(\omega) = \sum_{i=0}^{n-1} h_i(\omega)\big(M_{t_{i+1}\wedge t}(\omega) - M_{t_i\wedge t}(\omega)\big).$$

Path for $H \in b\mathcal{E}$



Stochastic Integral Process for $H \in b\mathcal{E}$



Extension to the Closure of $b\mathcal{E}$

Lemma 2

Suppose M is a square-integrable martingale. For every $H \in b\mathcal{E}$, $(H \cdot M) \in \mathcal{M}_0^2$, i.e. $(H \cdot M)$ is a square-integrable martingale null at 0 with $\sup_{t \geq 0} E\left[(H \cdot M)_t^2\right] < \infty$, and we have the isometry property

$$E\left[(H \cdot M)_\infty^2\right] = E\left[\left(\int_0^\infty H_s dM_s\right)^2\right] = E\left[\int_0^\infty H_s^2 d[M]_s\right].$$

- Isometry is a distance-preserving transformation between two normed spaces – what is the transformation and what are the norms?
- Transformation: $b\mathcal{E} \to \mathcal{M}_0^2$, $H \mapsto (H \cdot M)$ (for a fixed $M \in \mathcal{H}^2$).
- Norm on $b\mathcal{E}$: $\|H\|_{L^2(M)} := E\left[\int_0^\infty H_s^2 d[M]_s\right]^{1/2}$ (for a fixed $M \in \mathcal{H}^2$).
- Norm on $\mathcal{M}_0^2$: $\|(H \cdot M)\|_{\mathcal{M}_0^2} := E\left[(H \cdot M)_\infty^2\right]^{1/2}$ (for a fixed $M \in \mathcal{H}^2$).

Extension to the Closure of $b\mathcal{E}$

Theorem 3 (Continuous linear extension)

Every bounded linear transformation $T : X \to Y$ from a normed vector space X to a complete *normed vector space Y can be uniquely extended to a bounded linear transformation $\overline{T} : \overline{X} \to Y$ from closure of X to Y.*

- It turns out that $\mathcal{M}_0^2$ is complete in the $\|\cdot\|_{\mathcal{M}_0^2}$ norm, so we only need to know what is the closure of $b\mathcal{E}$.
- When $M \in \mathcal{M}_0^2$ the closure of $b\mathcal{E}$ in the $\|\cdot\|_{L^2(M)}$ is

$$L^2(M) := \left\{ \text{all predictable } H \text{ with } \|H\|_{L^2(M)} = E\left[\int_0^\infty H_s^2 d[M]_s\right]^{1/2} < \infty \right\},$$

which means that we can uniquely extend the definition of $(H \cdot M)$ to $H \in L^2(M)$ for $M \in \mathcal{M}_0^2$ and the properties of the map $b\mathcal{E} \to \mathcal{M}_0^2$, $H \mapsto (H \cdot M)$ are retained.

Extension to the Closure of $b\mathcal{E}$

The extension is done in the following way:

1. We take a sequence of $H^n \in b\mathcal{E}$ such that H^n converges to H in $L^2(M)$ norm for a fixed $M \in \mathcal{M}_0^2$, i.e.

$$\lim_{n\to\infty} \left\|H^n - H\right\|_{L^2(M)} = 0.$$

   Because the closure of $b\mathcal{E}$ is $L^2(M)$ (i.e. $b\mathcal{E}$ is dense in $L^2(M)$), such a sequence exists.

2. By the isometry property, we also have that $(H^n \cdot M)$ converges to $(H \cdot M)$ in $\mathcal{M}_0^2$ norm, i.e.

$$\lim_{n\to\infty} \left\|(H^n \cdot M) - (H \cdot M)\right\|_{\mathcal{M}_0^2} = 0.$$

3. We can therefore define

$$\left(H \cdot M\right)_t(\omega) := \left(\lim_{n\to\infty} (H^n \cdot M)\right)_t(\omega),$$

   where the limit is taken with respect to $\| \cdot \|_{\mathcal{M}_0^2}$.

Extension to Locally Square-Integrable Integrators

Problem: This definition does not admit Brownian motion W as an integrator. We have that $E\left[W_t^2\right] = t$, so $\sup_{t\geq 0} W_t^2 = \infty$, meaning that $W \notin \mathcal{M}_0^2$.

Definition 4 (Locally square-integrable local martingale)

A local martingale M null at zero is called *locally square-integrable* and we write $M \in \mathcal{M}_{0,\mathrm{loc}}^2$ if there exists a sequence of stopping times $\tau_n \nearrow \infty$ P-a.s. such that $M^{\tau_n} \in \mathcal{M}_0^2$ for each n.

Definition 5 ($L^2_{\mathrm{loc}}(M)$)

We say for a predictable process H that $H \in L^2_{\mathrm{loc}}(M)$ if there exists a sequence of stopping times $\tau_n \nearrow \infty$ P-a.s. such that $H\mathbb{1}_{]\!]0,\tau_n]\!]} \in L^2(M)$ for each n.

- When $H \in L^2_{\mathrm{loc}}(M)$ and $M \in \mathcal{M}_{0,\mathrm{loc}}^2$ we can define $(H \cdot M) := \left(H\mathbb{1}_{]\!]0,\tau_n]\!]} \cdot M^{\tau_n}\right)$ on $]\!]0,\tau_n]\!]$.
- Here $]\!]0,\tau_n]\!]$ denotes the so-called *stochastic interval* defined by

$$]\!]0,\tau_n]\!] := \{(\omega,t) \in \Omega \times \mathbb{R}_+ \mid 0 < t \leq \tau_n\}.$$

Extension to Local Martingales and Semimartingales

We do not go into details in this case, but it is necessary to know that the definition of $(H \cdot M)$ for $H \in L^2_{\text{loc}}(M)$ can be extended to a general local martingale M. This also motivates the extension to *semimartingales*.

Definition 6 (Semimartingale)

A *semimartingale* is a stochastic process $X = (X_t)_{t \geq 0}$ that can be decomposed as $X = X_0 + M + A$, where M is a local martingale null at 0 and A is an adapted RCLL process null at 0 with trajectories of finite variation. We say that X is a *special semimartingale* if A is even predictable.

- Since the stochastic integral is linear, the definition of a semimartingale offers itself for defining $(H \cdot X) := (H \cdot M) + (H \cdot A)$.
- A is of finite variation, so $(H \cdot A)$ can be defined ω-wise as Lebesgue-Stieltjes integral, i.e.

$$(H \cdot A)_t(\omega) := \int_0^t H_s(\omega) dA_s(\omega)$$

  This works well for predictable, locally bounded H.

Thank you for your attention!