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Mathematical Foundations for Finance

Exercise sheet 9

Exercise 9.1 Let $(Y_k)_{k\in\mathbb{N}}$ be a sequence of independent random variables defined on a probability space $(\Omega,\mathcal{F},P)$ and consider the filtration $\mathbb{F}=(\mathcal{F}_k)_{k\in\mathbb{N}_0}$ given by $\mathcal{F}_0=\{\emptyset,\Omega\}$ and $\mathcal{F}_k=\sigma(Y_1,\dots,Y_k)$ for all $k\in\mathbb{N}$. Let $E[Y_k]=\mu$ and $\mathrm{Var}(Y_k)=\sigma^2$ for all $k\in\mathbb{N}$ with $\mu\in\mathbb{R}$ and $\sigma^2>0$. Define additionally $X=(X_k)_{k\in\mathbb{N}_0}$ by

$$X_k=\sum_{j=1}^{k} Y_j \quad \text{for all } k\in\mathbb{N}_0,$$

and assume that X is adapted to $\mathbb{F}$.

(a) Show that for any $\mathbb{F}$-adapted integrable process $Z=(Z_k)_{k\in\mathbb{N}_0}$, there exists a P-a.s. unique decomposition of Z into $Z=M+A$ with $M=(M_k)_{k\in\mathbb{N}_0}$ a $(P,\mathbb{F})$-martingale and $A=(A_k)_{k\in\mathbb{N}_0}$ an $\mathbb{F}$-predictable integrable process with $A_0=0$.
Hint: This is the Doob decomposition. Show the existence by construction.

(b) Using (a), explicitly derive the processes M and A in the Doob decomposition of X.

(c) Explicitly derive the optional quadratic variation $[M]=\big([M]_k\big)_{k\in\mathbb{N}_0}$ of the square-integrable martingale M from (b), and show that $M^2-[M]$ is a martingale.
Hint: See Theorem V.1.1 in the lecture notes, and use that due to the condition $\Delta[M]=(\Delta M)^2$, *we must have that* $[M]_k-[M]_{k-1}=(M_k-M_{k-1})^2$.

(d) Explicitly derive the predictable compensator $\langle M\rangle=\big(\langle M\rangle_k\big)_{k\in\mathbb{N}_0}$ of the process M from (b).
Hint: See the remark on page 79 in the lecture notes. Also use that if M is a square-integrable martingale, then $\langle M\rangle$ is integrable.

Solution 9.1

(a) Let Z be an $\mathbb{F}$-adapted integrable process. Since we want that

$$Z_k-Z_{k-1}=M_k-M_{k-1}+A_k-A_{k-1} \quad \text{for all } k\in\mathbb{N}$$

and since we require for M that

$$E[M_k-M_{k-1}\,|\,\mathcal{F}_{k-1}]=0 \quad \text{for all } k\in\mathbb{N},$$

we must also have that

$$E[A_k-A_{k-1}\,|\,\mathcal{F}_{k-1}]=E[Z_k-Z_{k-1}\,|\,\mathcal{F}_{k-1}] \quad \text{for all } k\in\mathbb{N}.$$

However, since A is required to be predictable, this is equivalent to

$$A_k-A_{k-1}=E[Z_k-Z_{k-1}\,|\,\mathcal{F}_{k-1}] \quad \text{for all } k\in\mathbb{N}.$$

Together with the requirement that $A_0=0$, these increments determine A uniquely, giving us that

$$A_k=\sum_{j=1}^{k}\big(E[Z_j\,|\,\mathcal{F}_{j-1}]-Z_{j-1}\big).$$

We can then define $M_k := Z_k - A_k$, which gives both existence and uniqueness for M due to the uniqueness of A. We then obviously have that

$$M_k = Z_0 + \sum_{j=1}^{k} \big(Z_j - E\left[Z_j \,|\, \mathcal{F}_{j-1} \right] \big).$$

(b) We have seen in (a) that we must have

$$A_k - A_{k-1} = E\left[X_k - X_{k-1} \,|\, \mathcal{F}_{k-1} \right] = E\left[Y_k \,|\, \mathcal{F}_{k-1} \right] = E\left[Y_k \right] = \mu,$$

where the third equality follows from the fact that Y_k is independent of $\mathcal{F}_{k-1}$ by the definition of $\mathbb{F}$. But the above directly gives us that $A_k = k\mu$. Since $X_k = M_k + A_k$, this in turn gives that $M_k = X_k - k\mu$.

(c) Since the process M from (b) is a (square-integrable) martingale, Theorem V.1.1 from the lecture notes states that there exists a unique $\mathbb{F}$-adapted, increasing RCLL process $[M] = ([M]_k)_{k \in \mathbb{N}_0}$ null at 0 with $\Delta[M] = (\Delta M)^2$ and having a property that $M^2 - [M]$ is a local martingale. As noted in the hint, the requirement that $\Delta[M] = (\Delta M)^2$ translates to $[M]_k - [M]_{k-1} = (M_k - M_{k-1})^2$, which means that

$$[M]_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2. \tag{1}$$

This indeed is an $\mathbb{F}$-adapted increasing process null at 0. It is also integrable, since M is square-integrable. In order to show that $M^2 - [M]$ is a martingale, we therefore only need to show the martingale property, i.e. that

$$\begin{aligned} & E\left[M_k^2 - M_{k-1}^2 - ([M]_k - [M]_{k-1}) \,\middle|\, \mathcal{F}_{k-1} \right] = 0 \\ \iff \quad & E\left[M_k^2 - M_{k-1}^2 - (M_k - M_{k-1})^2 \,\middle|\, \mathcal{F}_{k-1} \right] = 0 \\ \iff \quad & E\left[2M_{k-1}(M_k - M_{k-1}) \,|\, \mathcal{F}_{k-1} \right] = 0 \\ \iff \quad & 2M_{k-1} E\left[M_k - M_{k-1} \,|\, \mathcal{F}_{k-1} \right] = 0. \end{aligned}$$

But the last equality holds since M is a martingale, so $M^2 - [M]$ is indeed a martingale.

When $M_k = X_k - k\mu$, we get from (1) that

$$[M]_k = \sum_{j=1}^{k} (Y_j - \mu)^2.$$

(d) As stated in the remark on page 79 in the lecture notes, since the process $[M]$ from (c) is integrable, there exists a unique increasing predictable process $\langle M \rangle = (\langle M \rangle_k)_{k \in \mathbb{N}_0}$ null at 0 such that $[M] - \langle M \rangle$ is a local martingale. Since M is even a square-integrable martingale, the hint tells us that $\langle M \rangle$ is integrable, which gives us the integrability condition for $[M] - \langle M \rangle$.

Let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence for $[M] - \langle M \rangle$. Then for all $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$, we have

$$E\left[[M]_{k \wedge \tau_n} \,|\, \mathcal{F}_{k-1} \right] - E\left[\langle M \rangle_{k \wedge \tau_n} \,|\, \mathcal{F}_{k-1} \right] = [M]_{(k-1) \wedge \tau_n} - \langle M \rangle_{(k-1) \wedge \tau_n}. \tag{2}$$

But both $[M]$ and $\langle M \rangle$ are increasing and positive, therefore we have that

$$\begin{aligned} |[M]_{k \wedge \tau_n}| &= [M]_{k \wedge \tau_n} \leq [M]_k \in L^1(P), \\ |\langle M \rangle_{k \wedge \tau_n}| &= \langle M \rangle_{k \wedge \tau_n} \leq \langle M \rangle_k \in L^1(P). \end{aligned}$$

When we thus take the limit of both sides in (2), we can use the dominated convergence theorem for conditional expectations to pull the limits inside the conditional expectation operators and we obtain that

$$E\left[[M]_k - \langle M\rangle_k \,|\, \mathcal{F}_{k-1}\right] = [M]_{k-1} - \langle M\rangle_{k-1}.$$

So $[M] - \langle M\rangle$ is even a martingale. Rearranging the above and using that $\langle M\rangle$ is predictable gives us that

$$\langle M\rangle_k - \langle M\rangle_{k-1} = E\left[[M]_k - [M]_{k-1} \,|\, \mathcal{F}_{k-1}\right] = E\left[(Y_k-\mu)^2 \,\middle|\, \mathcal{F}_{k-1}\right] = \mathrm{Var}(Y_k) = \sigma^2,$$

which is turn gives that $\langle M\rangle_k = k\sigma^2$.

Exercise 9.2 A *Poisson process* with parameter $\lambda > 0$ with respect to a probability measure P and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a (real-valued) stochastic process $N = (N_t)_{t\geq 0}$ which is adapted to $\mathbb{F}$, $N_0 = 0$ P-a.s. and satisfies the following two properties:

(PP1) For $0 \leq s < t$, the *increment* $N_t - N_s$ is independent (under P) of $\mathcal{F}_s$ and is (under P) *Poisson-distributed* with parameter $\lambda(t-s)$, i.e.

$$P[N_t - N_s = k] = \frac{(\lambda(t-s))^k}{k!}e^{-\lambda(t-s)}, \quad k \in \mathbb{N}_0.$$

(PP2) N is a *counting process* with jumps of size 1, i.e. for P-almost all $\omega \in \Omega$, the function $t \mapsto N_t(\omega)$ is right-continuous with left limits (RCLL), piecewise constant, $\mathbb{N}_0$-valued, and increases by jumps of size 1.

Poisson processes form the cornerstone of *jump processes*, which are of importance in advanced financial modeling. Show that the following processes are $(P, \mathbb{F})$-martingales:

(a) $\widetilde{N}_t := N_t - \lambda t$, $t \geq 0$. This process is also called a *compensated Poisson process.*
Hint: If $X \sim$ *Poi*(λ)*, then* $E[X] = \lambda$.

(b) $\widetilde{N}_t^2 - N_t$, $t \geq 0$, and $\widetilde{N}_t^2 - \lambda t$, $t \geq 0$. Use these results to derive $[\widetilde{N}]$ and $\langle \widetilde{N}\rangle$.
Hint: If $X \sim$ *Poi*(λ)*, then* $\mathrm{Var}(X) = \lambda$.

(c) $S_t := e^{N_t \log(1+\sigma) - \lambda\sigma t}$, $t \geq 0$, where $\sigma > -1$. S is also called a *geometric Poisson process.*

Solution 9.2 In all three cases, adaptedness is obvious and integrability is also clear, since each $N_t = N_t - N_0 \sim \mathrm{Poi}(\lambda t)$ has a Poisson distribution, which has finite exponential moments and hence also finite moments of all orders. What remains to be shown in all cases is the *martingale property.* Let $0 \leq s < t$.

(a) Using that $N_t - N_s \sim \mathrm{Poi}(\lambda(t-s))$ is independent of $\mathcal{F}_s$, we get

$$E[N_t - N_s \,|\, \mathcal{F}_s] = E[N_t - N_s] = \lambda(t-s) = \lambda t - \lambda s \quad P\text{-a.s.}$$

Since N_s is $\mathcal{F}_s$-measurable, we can rearrange the above equation to obtain

$$E[N_t - \lambda t \,|\, \mathcal{F}_s] = N_s - \lambda s \quad P\text{-a.s.},$$

which is what we wanted to show.

(b) We have that

$$\begin{aligned}E\left[\widetilde{N}_t^2-\widetilde{N}_s^2\,\middle|\,\mathcal{F}_s\right]&=E\left[\widetilde{N}_t^2-2\widetilde{N}_s\widetilde{N}_t+\widetilde{N}_s^2+2\widetilde{N}_s\widetilde{N}_t-2\widetilde{N}_s^2\,\middle|\,\mathcal{F}_s\right]\\&=E\left[(\widetilde{N}_t-\widetilde{N}_s)^2+2\widetilde{N}_s\widetilde{N}_t-2\widetilde{N}_s^2\,\middle|\,\mathcal{F}_s\right]\\&=E\left[(\widetilde{N}_t-\widetilde{N}_s)^2\,\middle|\,\mathcal{F}_s\right]+2\widetilde{N}_sE\left[\widetilde{N}_t-\widetilde{N}_s\,\middle|\,\mathcal{F}_s\right]\\&=E\left[(\widetilde{N}_t-\widetilde{N}_s)^2\,\middle|\,\mathcal{F}_s\right]=E\left[(N_t-N_s-\lambda(t-s))^2\,\middle|\,\mathcal{F}_s\right]\\&=E\left[(N_t-N_s-E\left[N_t-N_s\right])^2\right]\\&=\operatorname{Var}(N_t-N_s)=\lambda(t-s),\end{aligned}$$

where the second term in the third equality is equal to 0 since $\widetilde{N}$ is a martingale as shown in (a). Since $\widetilde{N}_s^2$ is $\mathcal{F}_s$-measurable, we can rearrange this to obtain that

$$E\left[\widetilde{N}_t^2-\lambda t\,\middle|\,\mathcal{F}_s\right]=\widetilde{N}_s^2-\lambda s,$$

which gives the martingale property for the process $(\widetilde{N}_t^2-\lambda t)_{t\geq 0}$.

Using the previous result, we can also easily compute that

$$\begin{aligned}E\left[\widetilde{N}_t^2-N_t-(\widetilde{N}_s^2-N_s)\,\middle|\,\mathcal{F}_s\right]&=E\left[\widetilde{N}_t^2-\widetilde{N}_s^2-(N_t-N_s)\,\middle|\,\mathcal{F}_s\right]\\&=E\left[\widetilde{N}_t^2-\widetilde{N}_s^2\,\middle|\,\mathcal{F}_s\right]-E\left[N_t-N_s\,|\,\mathcal{F}_s\right]\\&=\lambda(t-s)-\lambda(t-s)=0,\end{aligned}$$

giving the martingale property for the process $(\widetilde{N}_t^2-N_t)_{t\geq 0}$. In addition, N is null at zero, adapted to $\mathbb{F}$, increasing and we have that

$$\Delta N=(\Delta N)^2$$

because all jumps of N are of size 1. By Theorem V.1.1 we therefore have that $[\widetilde{N}]=N$. Additionally, the process $(\lambda t)_{t\geq 0}$ is null at 0, predictable, increasing, and we have that

$$N_t-\lambda t=[\widetilde{N}]_t-\lambda t$$

is a (local) martingale, which means that we have $\langle\widetilde{N}\rangle_t=\lambda t$.

(c) If $X\sim\operatorname{Poi}(\mu)$ and $a>0$, we have that

$$E\left[e^{aX}\right]=\sum_{k=0}^{\infty}e^{ak}\frac{\mu^k}{k!}e^{-\mu}=e^{-\mu}\sum_{k=0}^{\infty}\frac{(e^a\mu)^k}{k!}=e^{-\mu}e^{e^a\mu}=e^{\mu(e^a-1)}.$$

Using this result and the fact that $N_t-N_s\sim\operatorname{Poi}(\lambda(t-s))$ is independent of $\mathcal{F}_s$, we get

$$\begin{aligned}E\left[\frac{S_t}{S_s}\,\middle|\,\mathcal{F}_s\right]&=E\left[e^{(N_t-N_s)\log(1+\sigma)-\lambda\sigma(t-s)}\,\middle|\,\mathcal{F}_s\right]\\&=e^{-\lambda\sigma(t-s)}E\left[e^{(N_t-N_s)\log(1+\sigma)}\right]\\&=e^{-\lambda\sigma(t-s)}e^{\lambda(t-s)(1+\sigma-1)}=1\quad P\text{-a.s.}\end{aligned}$$

Exercise 9.3 Let $W=(W_t)_{t\geq 0}$ be a Brownian motion. Let $a,b>0$ and define

$$\begin{aligned}\tau_a&=\inf\{t\geq 0\,|\,W_t>a\},\\\sigma_{a,b}&=\inf\{t\geq 0\,|\,W_t>a+bt\}.\end{aligned}$$

(a) Show that for $\tau \in \{\tau_a, \sigma_{a,b}\}$ and all $\alpha \in \mathbb{R}$ we have that

$$E\left[e^{\alpha W_{\tau\wedge t} - \frac{1}{2}\alpha^2(\tau\wedge t)}\right] = 1.$$

(b) Using your result from (a) show that

$$e^{\alpha a} E\left[e^{-\frac{1}{2}\alpha^2 \tau_a}\right] = 1,$$

and use this to conclude by an appropriate choice of α that the Laplace transform ϕ_{τ_a} of τ_a is given by

$$\phi_{\tau_a}(\lambda) := E\left[e^{-\lambda\tau_a}\right] = e^{-a\sqrt{2\lambda}}, \quad \lambda > 0.$$

Hint 1: Make use of dominated convergence theorem.
Hint 2: Use that $W_{\tau_a} = a$ P-a.s.; we will show this in another exercise sheet.

(c) Using your result from (a) show that

$$e^{\alpha a} E\left[e^{(ab-\frac{1}{2}\alpha^2)\sigma_{a,b}}\right] = 1,$$

and use this to conclude by an appropriate choice of α that the Laplace transform $\phi_{\sigma_{a,b}}$ of $\sigma_{a,b}$ is given by

$$\phi_{\sigma_{a,b}}(\lambda) := E\left[e^{-\lambda\sigma_{a,b}}\right] = e^{-a(b+\sqrt{b^2+2\lambda})}, \quad \lambda > 0.$$

Hint 1: Make use of dominated convergence theorem.
Hint 2: Use that $W_{\sigma_{a,b}} = a + b\sigma_{a,b}$ P-a.s.

(d) Show that τ_a is P-a.s. finite for any $a > 0$ and that $\sigma_{a,b}$ takes the value of $+\infty$ with a positive probability for any $a, b > 0$.

Solution 9.3

(a) We know from Proposition IV.2.2 in the lecture notes that the process $M = (M_t)_{t\geq 0}$ given by

$$M_t = e^{\alpha W_t - \frac{1}{2}\alpha^2 t}$$

is a martingale for all $\alpha \in \mathbb{R}$. The stopping theorem (Theorem IV.2.1 in the lecture notes) then implies that for any stopping time τ, the stopped process M^τ is also a martingale, which gives that

$$1 = E[M_0] = E[M_0^\tau] = E[M_t^\tau] = E[M_{\tau\wedge t}] = E\left[e^{\alpha W_{\tau\wedge t} - \frac{1}{2}\alpha^2(\tau\wedge t)}\right],$$

since $\tau \wedge t$ is a bounded stopping time for all $t \geq 0$.

(b) We clearly have that $W_{\tau_a\wedge t} \leq a$ by the definition of τ_a. As a consequence, we have that $M_{\tau_a\wedge t} \leq e^{\alpha a}$ for all $t > 0$. Dominated convergence theorem therefore gives that

$$\begin{aligned} 1 &= \lim_{t\to\infty} E\left[e^{\alpha W_{\tau_a\wedge t} - \frac{1}{2}\alpha^2(\tau_a\wedge t)}\right] = E\left[\lim_{t\to\infty} e^{\alpha W_{\tau_a\wedge t} - \frac{1}{2}\alpha^2(\tau_a\wedge t)}\right] \\ &= E\left[e^{W_{\tau_a} - \frac{1}{2}\alpha^2\tau_a}\right] = e^{\alpha a} E\left[e^{-\frac{1}{2}\alpha^2\tau_a}\right], \end{aligned}$$

where the last equality follows from the fact that $W_{\tau_a} = a$ P-a.s. Reorganizing the above and setting $\alpha = \sqrt{2\lambda}$ for any $\lambda > 0$ gives that

$$E\left[e^{-\lambda\tau_a}\right] = e^{-a\sqrt{2\lambda}}, \quad \lambda > 0.$$

(c) We proceed analogously to (b). We have that $W_{\sigma_{a,b}\wedge t} \le a + b(\sigma_{a,b} \wedge t)$ by the definition of $\sigma_{a,b}$. As a consequence, we have that

$$M_{\sigma_{a,b}\wedge t} \le \exp\left(\alpha a + \left(\alpha b - \frac{1}{2}\alpha^2\right)(\sigma_{a,b} \wedge t)\right).$$

The right-hand side is not yet independent of t, but if we assume that

$$\alpha b < \frac{1}{2}\alpha^2 \quad \iff \quad \alpha > 2b,$$

then $M_{\sigma_{a,b}\wedge t} \le e^{\alpha a}$. So we can again apply dominated convergence theorem and obtain

$$\begin{aligned} 1 &= \lim_{t\to\infty} E\left[e^{\alpha W_{\sigma_{a,b}\wedge t} - \frac{1}{2}\alpha^2(\sigma_{a,b}\wedge t)}\right] = E\left[\lim_{t\to\infty} e^{\alpha W_{\sigma_{a,b}\wedge t} - \frac{1}{2}\alpha^2(\sigma_{a,b}\wedge t)}\right] = E\left[e^{\alpha W_{\sigma_{a,b}} - \frac{1}{2}\alpha^2 \sigma_{a,b}}\right] \\ &= E\left[e^{\alpha(a + b\sigma_{a,b}) - \frac{1}{2}\alpha^2\sigma_{a,b}}\right] = e^{\alpha a} E\left[e^{(\alpha b - \frac{1}{2}\alpha^2)\sigma_{a,b}}\right], \end{aligned}$$

where the fourth equality uses that $W_{\sigma_{a,b}} = a + b\sigma_{a,b}$. Reorganizing the above and setting $\alpha = b + \sqrt{b^2 + 2\lambda} > 2b$ for any $\lambda > 0$ gives that

$$E\left[e^{-\lambda\sigma_{a,b}}\right] = e^{-a(b+\sqrt{b^2+2\lambda})}.$$

(d) We have for any stopping time τ and any $\lambda > 0$ that

$$E\left[e^{-\lambda\tau}\right] = E\left[e^{-\lambda\tau}\mathbb{1}_{\{\tau<\infty\}} + e^{-\lambda\tau}\mathbb{1}_{\{\tau=\infty\}}\right] = E\left[e^{-\lambda\tau}\mathbb{1}_{\{\tau<\infty\}}\right].$$

Since $e^{-\lambda\tau} \le 1$ for all $\lambda > 0$, dominated convergence theorem gives that

$$\lim_{\lambda\downarrow 0} E\left[e^{-\lambda\tau}\mathbb{1}_{\{\tau<\infty\}}\right] = E\left[\lim_{\lambda\downarrow 0} e^{-\lambda\tau}\mathbb{1}_{\{\tau<\infty\}}\right] = E\left[\mathbb{1}_{\{\tau<\infty\}}\right] = P\left[\tau < \infty\right].$$

So we conclude that

$$\lim_{\lambda\downarrow 0} \phi_\tau(\lambda) = P\left[\tau < \infty\right].$$

Using the expressions derived for the Laplace transforms of τ_a and $\sigma_{a.b}$ from (b) and (c), we obtain

$$\begin{aligned} P\left[\tau_a < \infty\right] &= \lim_{\lambda\downarrow 0} e^{-a\sqrt{2\lambda}} = 1, \\ P\left[\sigma_{a,b} < \infty\right] &= \lim_{\lambda\downarrow 0} e^{-a(b+\sqrt{b^2+2\lambda})} = e^{-2ab} < 1. \end{aligned}$$

So while τ_a is P-a.s. finite for any $a > 0$, $\sigma_{a,b}$ takes the value of $+\infty$ with probability $1 - e^{-2ab}$.