# Non-Life Insurance: Mathematics and Statistics

## Exercise sheet 1

**Exercise 1.1 Discrete Distribution**
Suppose the random variable $N$ follows a geometric distribution with parameter $p \in (0,1)$, i.e.

$$\mathbb{P}[N=k] = \begin{cases} (1-p)^{k-1}p, & \text{if } k \in \mathbb{N}\setminus\{0\}, \\ 0, & \text{else.} \end{cases}$$

(a) Show that the geometric distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) Let $n \in \mathbb{N}\setminus\{0\}$. Calculate $\mathbb{P}[N \geq n]$.

(c) Calculate $\mathbb{E}[N]$.

(d) Let $r < -\log(1-p)$. Calculate $M_N(r) = \mathbb{E}[\exp\{rN\}]$.
Remark: $M_N$ is called the moment generating function of $N$.

(e) Calculate $\frac{d}{dr}M_N(r)\big|_{r=0}$. What do you observe?

**Exercise 1.2 Absolutely Continuous Distribution**
Suppose the random variable $Y$ follows an exponential distribution with parameter $\lambda > 0$, i.e. the density $f_Y$ of $Y$ is given by

$$f_Y(x) = \begin{cases} \lambda \exp\{-\lambda x\}, & \text{if } x \geq 0, \\ 0, & \text{else.} \end{cases}$$

(a) Show that the exponential distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) Let $0 < y_1 < y_2$. Calculate $\mathbb{P}[y_1 \leq Y \leq y_2]$.

(c) Calculate $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.

(d) Let $r < \lambda$. Calculate $\log M_Y(r) = \log \mathbb{E}[\exp\{rY\}]$.
Remark: $\log M_Y$ is called the cumulant generating function of $Y$.

(e) Calculate $\frac{d^2}{dr^2}\log M_Y(r)\big|_{r=0}$. What do you observe?

**Exercise 1.3 Chebychev's Inequality and Law of Large Numbers**
Suppose that an insurance company provides insurance against bike theft. In our model a bike gets stolen with a probability of 0.1, and in case of a theft the insurance company has to pay 1'000 CHF. We assume that we have $n$ independent and identically distributed (i.i.d.) risks $X_1, \ldots, X_n$ with

$$X_i = \begin{cases} 1'000, & \text{with probability } 0.1, \\ 0, & \text{with probability } 0.9, \end{cases}$$

for all $i = 1, \ldots, n$. In this exercise we are interested in the probability

$$p(n) \overset{\text{def}}{=} \mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq 0.1\mu\right]$$

of a deviation of the sample mean $\frac{1}{n}\sum_{i=1}^{n} X_i$ to the mean claim size $\mu = \mathbb{E}[X_1]$ of at least 10%, and how diversification effects this probability.

(a) Calculate $\mu$.

(b) Suppose that $n = 1$. Calculate $p(1)$.

(c) Suppose that $n = 1'000$. Calculate $p(1'000)$.

(d) Apply Chebychev's inequality to derive a minimum number $n$ of risks such that $p(n) < 0.01$.

(e) What can you say about $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i$?

**Exercise 1.4 Conditional Distribution**

Suppose that an insurance company distinguishes between small and large claims, where a claim is called large if it exceeds a fixed threshold $\theta > 0$. We use the random variable $I$ to indicate whether a claim is small or large, i.e. we have $I = 0$ for a small claim and $I = 1$ for a large claim. In particular, we model $\mathbb{P}[I = 0] = 1 - p$ and $\mathbb{P}[I = 1] = p$ for some $p \in (0, 1)$. Finally, we model the size of a claim that belongs to the large claims section by the random variable $Y$: given that we have a small claim, $Y$ is equal to 0 almost surely and, given that we have a large claim, $Y$ follows a Pareto distribution with threshold $\theta > 0$ and tail index $\alpha > 0$, i.e. the density $f_{Y|I=1}$ of $Y|I = 1$ is given by

$$f_{Y|I=1}(x) = \begin{cases} \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{-(\alpha+1)}, & \text{if } x \geq \theta, \\ 0, & \text{else.} \end{cases}$$

(a) Let $y > \theta$. Calculate $\mathbb{P}[Y \geq y]$.

(b) Calculate $\mathbb{E}[Y]$.