Non-Life Insurance: Mathematics and Statistics

Exercise sheet 2

Exercise 2.1 Gaussian Distribution
For a random variable X we write $X \sim \mathcal{N}(\mu, \sigma^2)$ if X follows a Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. The moment generating function M_X of $X \sim \mathcal{N}(\mu, \sigma^2)$ is given by

$$M_X(r) = \exp\left\{r\mu + \frac{r^2\sigma^2}{2}\right\}, \qquad \text{for all } r \in \mathbb{R}.$$

(a) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $a, b \in \mathbb{R}$. Show that

$$a + bX \sim \mathcal{N}(a + b\mu, b^2\sigma^2).$$

(b) Let $X_1, \ldots, X_n$ be independent with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for all $i \in \{1, \ldots, n\}$. Show that

$$\sum_{i=1}^n X_i \sim \mathcal{N}\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right).$$

Exercise 2.2 Maximum Likelihood and Hypothesis Test
Let $Y_1, \ldots, Y_n$ be claim amounts in CHF that an insurance company has to pay. We assume that $Y_1, \ldots, Y_n$ are independent random variables that all follow a log-normal distribution with the same unknown parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Then, by definition, $\log Y_1, \ldots, \log Y_n$ are independent Gaussian random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. Let $n = 8$ and suppose that we have the following observations for $\log Y_1, \ldots, \log Y_8$:

$$x_1 = 9, \qquad x_2 = 4, \qquad x_3 = 6, \qquad x_4 = 7, \qquad x_5 = 3, \qquad x_6 = 11, \qquad x_7 = 6, \qquad x_8 = 10.$$

(a) Write down the joint density $f_{\mu,\sigma^2}(x_1, \ldots, x_8)$ of $\log Y_1, \ldots, \log Y_8$.

(b) Calculate $\log f_{\mu,\sigma^2}(x_1, \ldots, x_8)$.

(c) Calculate $(\widehat{\mu}, \widehat{\sigma}^2) = \arg \max_{(\mu,\sigma^2) \in \mathbb{R} \times \mathbb{R}_{>0}} \log f_{\mu,\sigma^2}(x_1, \ldots, x_8)$.
Remark: Since the logarithm is a monotonically increasing function, $\widehat{\mu}$ and $\widehat{\sigma}^2$ are chosen such that the joint density of $\log Y_1, \ldots, \log Y_n$ at the given observations $x_1, \ldots, x_8$ is maximized. Hence, $\widehat{\mu}$ and $\widehat{\sigma}^2$ are called maximum likelihood estimates (MLE).

(d) Now suppose that we are interested in the mean μ of the logarithm of the claim amounts. An expert claims that $\mu = 6$. Perform a statistical test to test the null hypothesis

$$H_0 : \mu = 6$$

against the (two-sided) alternative hypothesis

$$H_1 : \mu \neq 6.$$

Exercise 2.3 $\mathcal{X}^2$-Distribution
Assume that X_k has a $\mathcal{X}^2$-distribution with $k \in \mathbb{N}$ degrees of freedom, i.e. X_k is absolutely continuous with density

$$f_{X_k}(x) = \begin{cases} \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} \exp\{-x/2\}, & \text{if } x \geq 0, \\ 0, & \text{else.} \end{cases}$$

(a) Let M_{X_k} be the moment generating function of X_k. Show that

$$M_{X_k}(r) = \frac{1}{(1-2r)^{k/2}}, \qquad \text{for } r < 1/2.$$

(b) Let $Z \sim \mathcal{N}(0,1)$. Show that $Z^2 \overset{(d)}{=} X_1$.

(c) Let $Z_1, \ldots, Z_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. Show that $\sum_{i=1}^k Z_i^2 \overset{(d)}{=} X_k$.

Exercise 2.4 Variance Decomposition
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ any sub-σ-algebra of $\mathcal{F}$. Show that

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|\mathcal{G})] + \mathrm{Var}(\mathbb{E}[X|\mathcal{G}]).$$