Non-Life Insurance: Mathematics and Statistics

Solution sheet 1

Solution 1.1 Discrete Distribution

(a) Note that N only takes values in $\mathbb{N} \setminus \{0\}$ and that $p \in (0,1)$. Hence, we calculate

$$\mathbb{P}[N \in \mathbb{R}] = \sum_{k=1}^{\infty} \mathbb{P}[N = k] = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p \sum_{k=0}^{\infty} (1-p)^k = p \frac{1}{1-(1-p)} = p \frac{1}{p} = 1,$$

from which we can conclude that the geometric distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) For $n \in \mathbb{N} \setminus \{0\}$ we get

$$\mathbb{P}[N \geq n] = \sum_{k=n}^{\infty} \mathbb{P}[N = k] = \sum_{k=n}^{\infty} (1-p)^{k-1} p = (1-p)^{n-1} p \sum_{k=0}^{\infty} (1-p)^k = (1-p)^{n-1},$$

where we used that $\sum_{k=0}^{\infty} (1-p)^k = \frac{1}{p}$, as was shown in (a).

(c) The expectation of a discrete random variable that takes values in $\mathbb{N} \setminus \{0\}$ can be calculated (if it exists) as

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} k \cdot \mathbb{P}[N = k].$$

Thus, we get

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p = \sum_{k=0}^{\infty} (k+1)(1-p)^k p = \sum_{k=0}^{\infty} k(1-p)^k p + \sum_{k=0}^{\infty} (1-p)^k p = (1-p)\mathbb{E}[N] + 1,$$

where we used that $\sum_{k=0}^{\infty} (1-p)^k p = 1$, as was shown in (a). We conclude that $\mathbb{E}[N] = \frac{1}{p}$.

(d) Let $r \in \mathbb{R}$. Then, we calculate

$$\begin{aligned}
\mathbb{E}[\exp\{rN\}] &= \sum_{k=1}^{\infty} \exp\{rk\} \cdot \mathbb{P}[N = k] \\
&= \sum_{k=1}^{\infty} \exp\{rk\} (1-p)^{k-1} p \\
&= p \exp\{r\} \sum_{k=1}^{\infty} [(1-p) \exp\{r\}]^{k-1} \\
&= p \exp\{r\} \sum_{k=0}^{\infty} [(1-p) \exp\{r\}]^{k}.
\end{aligned}$$

Since $(1-p)\exp\{r\}$ is strictly positive, the sum on the right hand side converges if and only if $(1-p)\exp\{r\} < 1$, which is equivalent to $r < -\log(1-p)$. Hence, $\mathbb{E}[\exp\{rN\}]$ exists if and only if $r < -\log(1-p)$, and in this case we have

$$M_N(r) = \mathbb{E}[\exp\{rN\}] = p \exp\{r\} \frac{1}{1-(1-p)\exp\{r\}} = \frac{p \exp\{r\}}{1-(1-p)\exp\{r\}}.$$

(e) For $r < -\log(1-p)$ we have

$$\begin{aligned}
\frac{d}{dr} M_N(r) &= \frac{d}{dr} \frac{p\exp\{r\}}{1-(1-p)\exp\{r\}} \\
&= \frac{p\exp\{r\}[1-(1-p)\exp\{r\}] + p\exp\{r\}(1-p)\exp\{r\}}{[1-(1-p)\exp\{r\}]^2} \\
&= \frac{p\exp\{r\}}{[1-(1-p)\exp\{r\}]^2}.
\end{aligned}$$

Hence, we get

$$\frac{d}{dr} M_N(r)\Big|_{r=0} = \frac{p\exp\{0\}}{[1-(1-p)\exp\{0\}]^2} = \frac{p}{[1-(1-p)]^2} = \frac{p}{p^2} = \frac{1}{p}.$$

We observe that $\frac{d}{dr} M_N(r)\big|_{r=0} = \mathbb{E}[N]$, which holds in general for all random variables for which the moment generating function exists in an interval around 0.

Solution 1.2 Absolutely Continuous Distribution

(a) We calculate

$$\mathbb{P}[Y \in \mathbb{R}] = \int_{-\infty}^{\infty} f_Y(x)\, dx = \int_0^{\infty} \lambda \exp\{-\lambda x\}\, dx = [-\exp\{-\lambda x\}]_0^{\infty} = [-0-(-1)] = 1,$$

from which we can conclude that the exponential distribution indeed defines a probability distribution on $\mathbb{R}$.

(b) For $0 < y_1 < y_2$ we calculate

$$\begin{aligned}
\mathbb{P}[y_1 \le Y \le y_2] &= \int_{y_1}^{y_2} f_Y(x)\, dx \\
&= \int_{y_1}^{y_2} \lambda \exp\{-\lambda x\}\, dx \\
&= [-\exp\{-\lambda x\}]_{y_1}^{y_2} \\
&= \exp\{-\lambda y_1\} - \exp\{-\lambda y_2\}.
\end{aligned}$$

(c) The expectation and the second moment of an absolutely continuous random variable can be calculated (if they exist) as

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} x f_Y(x)\, dx \qquad \text{and} \qquad \mathbb{E}[Y^2] = \int_{-\infty}^{\infty} x^2 f_Y(x)\, dx.$$

Thus, using partial integration, we get

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^{\infty} x\lambda \exp\{-\lambda x\}\, dx \\
&= [-x\exp\{-\lambda x\}]_0^{\infty} + \int_0^{\infty} \exp\{-\lambda x\}\, dx \\
&= 0 + \left[-\frac{1}{\lambda}\exp\{-\lambda x\}\right]_0^{\infty} \\
&= \frac{1}{\lambda}.
\end{aligned}$$

The variance $\mathrm{Var}(Y)$ can be calculated as

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \mathbb{E}[Y^2] - \frac{1}{\lambda^2}.$$

For the second moment $\mathbb{E}[Y^2]$ we get, again using partial integration,

$$\begin{aligned}
\mathbb{E}[Y^2] &= \int_0^\infty x^2 \lambda \exp\{-\lambda x\}\, dx \\
&= \left[-x^2 \exp\{-\lambda x\}\right]_0^\infty + \int_0^\infty 2x \exp\{-\lambda x\}\, dx \\
&= 0 + \frac{2}{\lambda}\mathbb{E}[Y] \\
&= \frac{2}{\lambda^2},
\end{aligned}$$

from which we can conclude that

$$\mathrm{Var}(Y) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Note that for the exponential distribution both the expectation and the variance exist. The reason is that $\exp\{-\lambda x\}$ goes much faster to 0 than x or x^2 go to infinity, for all $\lambda > 0$.

(d) Let $r \in \mathbb{R}$. Then, we calculate

$$\mathbb{E}[\exp\{rY\}] = \int_0^\infty \exp\{rx\}\lambda \exp\{-\lambda x\}\, dx = \int_0^\infty \lambda \exp\{(r-\lambda)x\}\, dx.$$

The integral on the right hand side and therefore also $\mathbb{E}[\exp\{rY\}]$ exist if and only if $r < \lambda$. In this case we have

$$M_Y(r) = \mathbb{E}[\exp\{rY\}] = \frac{\lambda}{r-\lambda}\left[\exp\{(r-\lambda)x\}\right]_0^\infty = \frac{\lambda}{r-\lambda}(0-1) = \frac{\lambda}{\lambda - r}$$

and therefore

$$\log M_Y(r) = \log\left(\frac{\lambda}{\lambda - r}\right).$$

(e) For $r < \lambda$ we have

$$\frac{d^2}{dr^2}\log M_Y(r) = \frac{d^2}{dr^2}\log\left(\frac{\lambda}{\lambda - r}\right) = \frac{d^2}{dr^2}[\log(\lambda) - \log(\lambda - r)] = \frac{d}{dr}\frac{1}{\lambda - r} = \frac{1}{(\lambda - r)^2}.$$

Hence, we get

$$\frac{d^2}{dr^2}\log M_Y(r)|_{r=0} = \frac{1}{(\lambda - 0)^2} = \frac{1}{\lambda^2}.$$

We observe that $\frac{d^2}{dr^2}\log M_Y(r)|_{r=0} = \mathrm{Var}(Y)$, which holds in general for all random variables for which the moment generating function exists in an interval around 0.

Solution 1.3 Chebychev's Inequality and Law of Large Numbers

(a) We have $\mu = \mathbb{E}[X_1] = 1'000 \cdot 0.1 = 100$, and $0.1\mu = 10$.

(b) For $n = 1$ we get

$$\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| = |X_1 - 100| = \begin{cases} 900, & \text{with probability } 0.1, \\ 100, & \text{with probability } 0.9. \end{cases}$$

As both values 900 and 100 are bigger than 10, we conclude that $p(1) = 1$. In particular, if we only have $n = 1$ risk in our portfolio, then with probability 1 the corresponding claim amount deviates from the mean claim size by at least 10%.

(c) For $n \in \mathbb{N}$ we can write

$$\begin{aligned} p(n) &= \mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq 0.1\mu\right] = 1 - \mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| < 0.1\mu\right] \\ &= 1 - \mathbb{P}\left[-0.1\mu < \frac{1}{n}\sum_{i=1}^{n} X_i - \mu < 0.1\mu\right] = 1 - \mathbb{P}\left[0.9n\mu < \sum_{i=1}^{n} X_i < 1.1n\mu\right] \end{aligned}$$

For $n = 1'000$ we get

$$\begin{aligned} p(1'000) &= 1 - \mathbb{P}\left[90'000 < \sum_{i=1}^{1'000} X_i < 110'000\right] \\ &= 1 - \mathbb{P}\left[90 < \text{number of bikes stolen} < 110\right] \\ &= 1 - \sum_{k=91}^{109} \binom{1'000}{k} (0.1)^k (0.9)^{1'000-k} \\ &\approx 0.32. \end{aligned}$$

Thus, if we have $n = 1'000$ risks in our portfolio, then with probability 0.32 the sample mean of the claim amounts deviates from the mean claim size by at least 10%. In particular, diversification leads to a reduction of this probability.

(d) As

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[X_i\right] = \mathbb{E}[X_1] = \mu,$$

Chebyshev's inequality leads to

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq 0.1\mu\right] \leq \frac{\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)}{(0.1\mu)^2}.$$

Using the independence of $X_1, \ldots, X_n$, we get

$$\begin{aligned} \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) &= \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}\left(X_i\right) = \frac{1}{n}\mathrm{Var}(X_1) = \frac{1}{n}\mathbb{E}\left[(X_1 - \mu)^2\right] = \frac{1}{n}\left(900^2 \cdot 0.1 + 100^2 \cdot 0.9\right) \\ &= \frac{90'000}{n}. \end{aligned}$$

This implies

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq 0.1\mu\right] \leq \frac{90'000}{n(0.1\mu)^2} = \frac{900}{n}.$$

We have

$$\frac{900}{n} < 0.1 \qquad \Longleftrightarrow \qquad n > 9'000.$$

This implies that Chebyshev's inequality guarantees that if we have more than 9'000 risks, then the probability that the sample mean of the claim amounts deviates from the mean claim size by at least 10% is smaller than 1%.

(e) We have that $X_1, X_2, ...$ are i.i.d. and that $\mathbb{E}[|X_1|] = \mathbb{E}[X_1] = \mu < \infty$. Thus, we can apply the strong law of large numbers, and we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \longrightarrow \mathbb{E}[X_1] = \mu = 10, \quad \mathbb{P} - \text{a.s.}$$

Solution 1.4 Conditional Distribution

(a) For $y > \theta > 0$ we get

$$\begin{aligned}
\mathbb{P}[Y \geq y] &= \mathbb{P}[Y \geq y, I = 0] + \mathbb{P}[Y \geq y, I = 1] \\
&= \mathbb{P}[Y \geq y | I = 0]\mathbb{P}[I = 0] + \mathbb{P}[Y \geq y | I = 1]\mathbb{P}[I = 1] \\
&= 0 \cdot (1 - p) + \mathbb{P}[Y \geq y | I = 1] \cdot p \\
&= p \cdot \mathbb{P}[Y \geq y | I = 1],
\end{aligned}$$

since $Y|I = 0$ is equal to 0 almost surely and thus $\mathbb{P}[Y \geq y | I = 0] = 0$. Since $Y|I = 1 \sim$ Pareto(θ, α), we can calculate

$$\mathbb{P}[Y \geq y | I = 1] = \int_y^\infty f_{Y|I=1}(x)\, dx = \int_y^\infty \frac{\alpha}{\theta} \left(\frac{x}{\theta}\right)^{-(\alpha+1)} dx = \left[-\left(\frac{x}{\theta}\right)^{-\alpha} \right]_y^\infty = \left(\frac{y}{\theta}\right)^{-\alpha}.$$

We conclude that

$$\mathbb{P}[Y \geq y] = p \left(\frac{y}{\theta}\right)^{-\alpha}.$$

(b) Using that $Y|I = 0$ is equal to 0 almost surely and thus $\mathbb{E}[Y|I = 0] = 0$, we get

$$\mathbb{E}[Y] = \mathbb{E}[Y \cdot 1_{\{I=0\}}] + \mathbb{E}[Y \cdot 1_{\{I=1\}}] = \mathbb{E}[Y|I = 0]\mathbb{P}[I = 0] + \mathbb{E}[Y|I = 1]\mathbb{P}[I = 1] = p \cdot \mathbb{E}[Y|I = 1].$$

Since $Y \mid I = 1 \sim$ Pareto(θ, α), we can calculate

$$\mathbb{E}[Y|I = 1] = \int_{-\infty}^\infty x f_{Y|I=1}(x)\, dx = \int_\theta^\infty x \frac{\alpha}{\theta} \left(\frac{x}{\theta}\right)^{-(\alpha+1)} dx = \alpha\theta^\alpha \int_\theta^\infty x^{-\alpha}\, dx$$

We see that the integral on the right hand side and therefore also $\mathbb{E}[Y]$ exist if and only if $\alpha > 1$. In this case we get

$$\mathbb{E}[Y|I = 1] = \alpha\theta^\alpha \left[-\frac{1}{\alpha - 1} x^{-(\alpha-1)} \right]_\theta^\infty = \alpha\theta^\alpha \frac{1}{\alpha - 1} \theta^{-(\alpha-1)} = \theta \frac{\alpha}{\alpha - 1}.$$

We conclude that, if $\alpha > 1$, we get

$$\mathbb{E}[Y] = p\theta \frac{\alpha}{\alpha - 1}.$$

If $0 < \alpha \leq 1$, $\mathbb{E}[Y]$ does not exist.