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Non-Life Insurance: Mathematics and Statistics

Solution sheet 12

Solution 12.1 Chain-Ladder Algorithm

(a) According to formula (9.5) of the lecture notes, the CL factor f_j can be estimated by

$$\widehat{f}_j^{CL} = \frac{\sum_{i=1}^{I-j-1} C_{i,j+1}}{\sum_{i=1}^{I-j-1} C_{i,j}} = \sum_{i=1}^{I-j-1} \frac{C_{i,j}}{\sum_{n=1}^{I-j-1} C_{n,j}} \, \frac{C_{i,j+1}}{C_{i,j}},$$

for all $j \in \{0, \dots, J-1\}$. Then, for all $i \in \{2, \dots, I\}$ and $j \in \{1, \dots, J\}$ with $i + j > I$, $C_{i,j}$ can be predicted by

$$\widehat{C}_{i,j}^{CL} = C_{i,I-i} \prod_{k=I-i}^{j-1} \widehat{f}_k^{CL}.$$

In particular, for the prediction $\widehat{C}_{i,J}^{CL}$ of the ultimate claim $C_{i,J}$ we have

$$\widehat{C}_{i,J}^{CL} = C_{i,I-i} \prod_{j=I-i}^{J-1} \widehat{f}_j^{CL}. \tag{1}$$

The estimates $\widehat{f}_0^{CL}, \dots, \widehat{f}_{J-1}^{CL}$ and the prediction of the lower triangle $\mathcal{D}_I^c$ are then given by

accident	development year j									
year i	0	1	2	3	4	5	6	7	8	9
1										
2										10'663'318
3									10'646'884	10'662'008
4								9'734'574	9'744'764	9'758'606
5							9'837'277	9'847'906	9'858'214	9'872'218
6						10'005'044	10'056'528	10'067'393	10'077'931	10'092'247
7					9'419'776	9'485'469	9'534'279	9'544'580	9'554'571	9'568'143
8				8'445'057	8'570'389	8'630'159	8'674'568	8'683'940	8'693'030	8'705'378
9			8'243'496	8'432'051	8'557'190	8'616'868	8'661'208	8'670'566	8'679'642	8'691'971
10		8'470'989	9'129'696	9'338'521	9'477'113	9'543'206	9'592'313	9'602'676	9'612'728	9'626'383
$\widehat{f}_j^{CL}$	1.493	1.078	1.023	1.015	1.007	1.005	1.001	1.001	1.001	

Table 1: Estimates $\widehat{f}_0^{CL}, \dots, \widehat{f}_{J-1}^{CL}$ and prediction of the lower triangle $\mathcal{D}_I^c$.

We see that $\widehat{f}_0^{CL} \approx 1.5$, while $\widehat{f}_j^{CL}$ is close to 1, for all $j \in \{1, \dots, J-1\}$, i.e. we observe a rather fast claims settlement in this example.

(b) The CL reserves $\widehat{\mathcal{R}}_i^{CL}$ at time $t = I$ are given by

$$\widehat{\mathcal{R}}_i^{CL} = \widehat{C}_{i,J}^{CL} - C_{i,I-i} = C_{i,I-i} \left(\prod_{j=I-i}^{J-1} \widehat{f}_j^{CL} - 1 \right),$$

for all accident years $i \in \{2, \dots, I\}$. Moreover, since $C_{1,J} = C_{1,I-1}$ is known, we have $\widehat{\mathcal{R}}_1^{CL} = 0$.

Summarizing, we get the following CL reserves $\widehat{\mathcal{R}}_i^{CL}$:

accident year i	1	2	3	4	5	6	7	8	9	10
CL reserves $\widehat{\mathcal{R}}_i^{CL}$	0	15'126	26'257	34'538	85'302	156'494	286'121	449'167	1'043'242	3'950'815

Table 2: CL reserves $\widehat{\mathcal{R}}_i^{CL}$ for all accident years $i \in \{1, \dots, I\}$.

By aggregating the CL reserves over all accident years, we get the CL predictor $\widehat{\mathcal{R}}^{CL}$ for the outstanding loss liabilities of past exposure claims:

$$\widehat{\mathcal{R}}^{CL} = \sum_{i=1}^{I} \widehat{\mathcal{R}}_i^{CL} = 6\text{'}047\text{'}061.$$

Solution 12.2 Bornhuetter-Ferguson Algorithm

(a) For all $j \in \{0, \dots, J-1\}$ we define $\widehat{\beta}_j^{CL}$ as the proportion paid after the first j development periods according to the estimated CL pattern, i.e.

$$\widehat{\beta}_0^{CL} = \frac{1}{\prod_{l=0}^{J-1} \widehat{f}_l^{CL}} = \prod_{l=0}^{J-1} \frac{1}{\widehat{f}_l^{CL}}$$

and

$$\widehat{\beta}_j^{CL} = \frac{\prod_{l=0}^{j-1} \widehat{f}_l^{CL}}{\prod_{l=0}^{J-1} \widehat{f}_l^{CL}} = \prod_{l=j}^{J-1} \frac{1}{\widehat{f}_l^{CL}},$$

for all $j \in \{1, \dots, J-1\}$. We get the following proportions:

development period j	0	1	2	3	4	5	6	7	8
proportion $\widehat{\beta}_j^{CL}$ paid so far	0.590	0.880	0.948	0.970	0.984	0.991	0.996	0.998	0.999

Table 3: Proportions $\widehat{\beta}_j^{CL}$ paid after the first j development periods according to the estimated CL pattern.

According to formula (9.8) of the lecture notes, in the Bornhuetter-Ferguson method the ultimate claim $C_{i,J}$ is predicted by

$$\widehat{C}_{i,J}^{BF} = C_{i,I-i} + \widehat{\mu}_i \left(1 - \widehat{\beta}_{I-i}^{CL}\right),$$

for all accident years $i \in \{2, \dots, I\}$. Thus, the Bornhuetter-Ferguson reserves $\widehat{\mathcal{R}}_i^{BF}$ are given by

$$\widehat{\mathcal{R}}_i^{BF} = \widehat{C}_{i,J}^{BF} - C_{i,I-i} = \widehat{\mu}_i \left(1 - \widehat{\beta}_{I-i}^{CL}\right),$$

for all accident years $i \in \{2, \dots, I\}$. Moreover, since $C_{1,J} = C_{1,I-1}$ is known, we have $\widehat{\mathcal{R}}_1^{BF} = 0$.

Summarizing, we get the following BF reserves $\widehat{\mathcal{R}}_i^{BF}$:

accident year i	1	2	3	4	5	6	7	8	9	10
BF reserves $\widehat{\mathcal{R}}_i^{BF}$	0	16'124	26'998	37'575	95'434	178'024	341'305	574'089	1'318'646	4'768'384

Table 4: BF reserves $\widehat{\mathcal{R}}_i^{BF}$ for all accident years $i \in \{1, \ldots, I\}$.

By aggregating the BF reserves over all accident years, we get the BF predictor $\widehat{\mathcal{R}}^{BF}$ for the outstanding loss liabilities of past exposure claims:

$$\widehat{\mathcal{R}}^{BF} = \sum_{i=1}^{I} \widehat{\mathcal{R}}_i^{BF} = 7'356'580.$$

(b) Note that for accident year $i = 1$ we have

$$\widehat{\mathcal{R}}_1^{CL} = 0 = \widehat{\mathcal{R}}_1^{BF}.$$

Now let $i \in \{2, \ldots, I\}$. Then, we observe that

$$\widehat{\mathcal{R}}_i^{CL} < \widehat{\mathcal{R}}_i^{BF}.$$

This can be explained as follows: Equation (1) can be rewritten as

$$\begin{aligned}
\widehat{C}_{i,J}^{CL} &= C_{i,I-i} \prod_{j=I-i}^{J-1} \widehat{f}_j^{CL} \\
&= C_{i,I-i} + C_{i,I-i} \left(\prod_{j=I-i}^{J-1} \widehat{f}_j^{CL} - 1 \right) \\
&= C_{i,I-i} + C_{i,I-i} \prod_{j=I-i}^{J-1} \widehat{f}_j^{CL} \left(1 - \prod_{j=I-i}^{J-1} \frac{1}{\widehat{f}_j^{CL}} \right) \\
&= C_{i,I-i} + \widehat{C}_{i,J}^{CL} \left(1 - \widehat{\beta}_{I-i}^{CL} \right).
\end{aligned}$$

Comparing this to

$$\widehat{C}_{i,J}^{BF} = C_{i,I-i} + \widehat{\mu}_i \left(1 - \widehat{\beta}_{I-i}^{CL} \right)$$

and noting that for the prior information $\widehat{\mu}_i$ we have $\widehat{\mu}_i > \widehat{C}_{i,J}^{CL}$, we immediately see that

$$\widehat{C}_{i,J}^{CL} < \widehat{C}_{i,J}^{BF},$$

which of course implies that

$$\widehat{\mathcal{R}}_i^{CL} = \widehat{C}_{i,J}^{CL} - C_{i,I-i} < \widehat{C}_{i,J}^{BF} - C_{i,I-i} = \widehat{\mathcal{R}}_i^{BF}.$$

Concluding, we found that choosing a prior information $\widehat{\mu}_i$ bigger than the estimated CL ultimate $\widehat{C}_{i,J}^{CL}$ leads to more conservative, i.e. higher reserves in the Bornhuetter-Ferguson method compared to the chain-ladder method.

Solution 12.3 Mack's Formula and Merz-Wüthrich (MW) Formula

The R code used in this exercise is provided below. We get the following results:

accident year i	CL reserve $\widehat{\mathcal{R}}_i^{CL}$	$\sqrt{\text{total msep}}$ (Mack)	in % of the reserves	$\sqrt{\text{CDR msep}}$ (MW)	in % of the $\sqrt{\text{total msep}}$
1	0	–	–	–	–
2	15'126	267	1.8 %	267	100 %
3	26'257	914	3.5 %	884	97 %
4	34'538	3'058	8.9 %	2'948	96 %
5	85'302	7'628	8.9 %	7'018	92 %
6	156'494	33'341	21.3 %	32'470	97 %
7	286'121	73'467	25.7 %	66'178	90 %
8	449'167	85'398	19.0 %	50'296	59 %
9	1'043'242	134'337	12.9 %	104'311	78 %
10	3'950'815	410'817	10.4 %	385'773	94 %
total	6'047'061	462'960	7.7 %	420'220	91 %

Table 5: CL reserves $\widehat{\mathcal{R}}_i^{CL}$, Mack's square-rooted conditional mean square errors of prediction and MW's square-rooted conditional mean square errors of prediction for all accident years $i \in \{1, \ldots, I\}$.

(a) Mack's square-rooted conditional mean square errors of prediction give us confidence bounds around the estimated CL reserves. We see that for the total claims reserves the one standard deviation confidence bounds are 7.7%. The biggest uncertainties can be found for accident years 6, 7 and 8, where the one standard deviation confidence bounds are roughly 20% or even higher.

(b) MW's square-rooted conditional mean square errors of prediction measure the contribution of the next accounting year to the total (run-off) uncertainty given by Mack's square-rooted conditional mean square errors of prediction. For aggregated accident years, we see that 91% of the total uncertainty is due to the next accounting year. This high value can be explained by the fast claims settlement already noticed in Exercise 12.1, (a).

```
### Load the required packages
library(readxl)
library(ChainLadder)

### Download the data from the link indicated on the exercise sheet
### Store the data under the name "Exercise.12.Data.xls" in the same folder as this R-Code
### Load the data
data <- read_excel("Exercise.12.Data.xls", sheet = "Data_1", range = "B22:K31", col_names = FALSE)

### Bring the data in the appropriate triangular form and label the axes
tri <- as.triangle(as.matrix(data))
dimnames(tri)=list(origin=1:nrow(tri),dev=1:ncol(tri))

### Calculate the CL reserves and the corresponding mseps
```

```
M <- MackChainLadder(tri, est.sigma = "Mack")

### CL factors
M$f

### Full triangle
M$FullTriangle

### CL reserves and Mack's square-rooted mseps (including illustrations)
M
plot(M)
plot(M, lattice = TRUE)

### CL reserves, MW's square-rooted mseps and Mack's square-rooted mseps
CDR(M)

### Mack's square-rooted mseps in % of the reserves
round(CDR(M)[,3] / CDR(M)[,1],3) * 100

### MW's square-rooted mseps in % of Mack's square-rooted mseps
round(CDR(M)[,2] / CDR(M)[,3],2) * 100

### Full uncertainty picture
CDR(M, dev="all")
```

Solution 12.4 Conditional MSEP and Claims Development Result

Note that the equalities in this exercise involving a conditional expectation are to be understood in an almost sure sense.

(a) Since X is square-integrable, also $\mathbb{E}[X|\mathcal{D}]$ is. Now, by subtracting and adding $\mathbb{E}[X|\mathcal{D}]$, we can write

$$\begin{aligned}
\mathrm{msep}_{X|\mathcal{D}}\left(\widehat{X}\right) &= \mathbb{E}\left[\left(X-\widehat{X}\right)^2 \middle| \mathcal{D}\right] \\
&= \mathbb{E}\left[\left(X-\mathbb{E}[X|\mathcal{D}]+\mathbb{E}[X|\mathcal{D}]-\widehat{X}\right)^2 \middle| \mathcal{D}\right] \\
&= \mathbb{E}\left[\left(X-\mathbb{E}[X|\mathcal{D}]\right)^2 \middle| \mathcal{D}\right] + \mathbb{E}\left[\left(\mathbb{E}[X|\mathcal{D}]-\widehat{X}\right)^2 \middle| \mathcal{D}\right] \\
&\qquad + 2\mathbb{E}\left[\left(X-\mathbb{E}[X|\mathcal{D}]\right)\left(\mathbb{E}[X|\mathcal{D}_I]-\widehat{X}\right) \middle| \mathcal{D}\right].
\end{aligned}$$

By definition of the conditional variance, we have

$$\mathbb{E}\left[\left(X-\mathbb{E}[X|\mathcal{D}]\right)^2 \middle| \mathcal{D}\right] = \mathrm{Var}(X|\mathcal{D}).$$

Moreover, since $\mathbb{E}[X|\mathcal{D}]$ and $\widehat{X}$ are $\mathcal{D}$-measurable, we get

$$\mathbb{E}\left[\left(\mathbb{E}[X|\mathcal{D}] - \widehat{X}\right)^2 \middle| \mathcal{D}\right] = \left(\mathbb{E}[X|\mathcal{D}] - \widehat{X}\right)^2$$

and

$$\begin{aligned}\mathbb{E}\left[(X - \mathbb{E}[X|\mathcal{D}])\left(\mathbb{E}[X|\mathcal{D}] - \widehat{X}\right) \middle| \mathcal{D}\right] &= \left(\mathbb{E}[X|\mathcal{D}] - \widehat{X}\right)\mathbb{E}\left[(X - \mathbb{E}[X|\mathcal{D}]) \middle| \mathcal{D}\right] \\ &= \left(\mathbb{E}[X|\mathcal{D}] - \widehat{X}\right)(\mathbb{E}[X|\mathcal{D}] - \mathbb{E}[X|\mathcal{D}]) \\ &= 0.\end{aligned}$$

By collecting the terms, we get the desired result

$$\mathrm{msep}_{X|\mathcal{D}}\left(\widehat{X}\right) = \mathbb{E}\left[\left(X - \widehat{X}\right)^2 \middle| \mathcal{D}\right] = \mathrm{Var}(X|\mathcal{D}) + \left(\mathbb{E}[X|\mathcal{D}] - \widehat{X}\right)^2.$$

(b) For $t \in \mathbb{N}$ with $t \geq I$ and $i > t - J$, the claims development result $\mathrm{CDR}_{i,t+1}$ is defined by

$$\mathrm{CDR}_{i,t+1} = \widehat{C}_{i,J}^{(t)} - \widehat{C}_{i,J}^{(t+1)} = \mathbb{E}\left[C_{i,J}|\mathcal{D}_t\right] - \mathbb{E}\left[C_{i,J}|\mathcal{D}_{t+1}\right],$$

see formulas (9.27) and (9.29) of the lecture notes. The inclusion $\mathcal{D}_t \subset \mathcal{D}_{t+1}$ implies that $\mathrm{CDR}_{i,t+1}$ is $\mathcal{D}_{t+1}$-measurable. Moreover, using the tower property of conditional expectation, we get

$$\begin{aligned}\mathbb{E}\left[\mathrm{CDR}_{i,t+1}|\mathcal{D}_t\right] &= \mathbb{E}\left[\mathbb{E}\left[C_{i,J}|\mathcal{D}_t\right] - \mathbb{E}\left[C_{i,J}|\mathcal{D}_{t+1}\right]|\mathcal{D}_t\right] \\ &= \mathbb{E}\left[C_{i,J}|\mathcal{D}_t\right] - \mathbb{E}\left[C_{i,J}|\mathcal{D}_t\right] \\ &= 0.\end{aligned}$$

Note that this result is given in Corollary 9.13 of the lecture notes. In particular, it implies that

$$\mathbb{E}\left[\mathrm{CDR}_{i,t+1}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathrm{CDR}_{i,t+1}|\mathcal{D}_t\right]\right] = 0.$$

Since $t_1 < t_2$ by assumption, CDR_{i,t_1+1} is $\mathcal{D}_{t_2}$-measurable. Thus, we get

$$\begin{aligned}\mathbb{E}\left[\mathrm{CDR}_{i,t_1+1}\mathrm{CDR}_{i,t_2+1}\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathrm{CDR}_{i,t_1+1}\mathrm{CDR}_{i,t_2+1}|\mathcal{D}_{t_2}\right]\right] \\ &= \mathbb{E}\left[\mathrm{CDR}_{i,t_1+1}\mathbb{E}\left[\mathrm{CDR}_{i,t_2+1}|\mathcal{D}_{t_2}\right]\right] \\ &= \mathbb{E}\left[\mathrm{CDR}_{i,t_1+1} \cdot 0\right] \\ &= 0.\end{aligned}$$

We can conclude that

$$\mathrm{Cov}\left(\mathrm{CDR}_{i,t_1+1}, \mathrm{CDR}_{i,t_2+1}\right) = \mathbb{E}\left[\mathrm{CDR}_{i,t_1+1}\mathrm{CDR}_{i,t_2+1}\right] - \mathbb{E}\left[\mathrm{CDR}_{i,t_1+1}\right]\mathbb{E}\left[\mathrm{CDR}_{i,t_2+1}\right] = 0.$$