Non-Life Insurance: Mathematics and Statistics

Solution sheet 2

Solution 2.1 Gaussian Distribution

(a) The moment generating function of $a + bX$ can be calculated as

$$M_{a+bX}(r) = \mathbb{E}\left[\exp\left\{r(a+bX)\right\}\right] = \exp\left\{ra\right\}\mathbb{E}\left[\exp\left\{rbX\right\}\right] = \exp\left\{ra\right\} M_X(rb),$$

for all $r \in \mathbb{R}$. Using the formula for the moment generating function of X given on the exercise sheet, we get

$$M_{a+bX}(r) = \exp\left\{ra\right\}\ \exp\left\{rb\mu + \frac{(rb)^2\sigma^2}{2}\right\} = \exp\left\{r(a+b\mu) + \frac{r^2b^2\sigma^2}{2}\right\},$$

which is equal to the moment generating function of a Gaussian random variable with expectation $a + b\mu$ and variance $b^2\sigma^2$. Since the moment generating function uniquely determines the distribution, we conclude that

$$a + bX \sim \mathcal{N}(a + b\mu, b^2\sigma^2).$$

(b) Using the independence of $X_1, \ldots, X_n$, the moment generating function of $Y = \sum_{i=1}^n X_i$ can be calculated as

$$M_Y(r) = \mathbb{E}\left[\exp\left\{rY\right\}\right] = \mathbb{E}\left[\exp\left\{r\sum_{i=1}^n X_i\right\}\right] = \prod_{i=1}^n \mathbb{E}\left[\exp\left\{rX_i\right\}\right] = \prod_{i=1}^n M_{X_i}(r),$$

for all $r \in \mathbb{R}$. Using the formula for the moment generating function of a Gaussian random variable given on the exercise sheet, we get

$$M_Y(r) = \prod_{i=1}^n \exp\left\{r\mu_i + \frac{r^2\sigma_i^2}{2}\right\} = \exp\left\{r\sum_{i=1}^n \mu_i + \frac{r^2\sum_{i=1}^n \sigma_i^2}{2}\right\},$$

which is equal to the moment generating function of a Gaussian random variable with expectation $\sum_{i=1}^n \mu_i$ and variance $\sum_{i=1}^n \sigma_i^2$. Since the moment generating function uniquely determines the distribution, we conclude that

$$\sum_{i=1}^n X_i \sim \mathcal{N}\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right).$$

Solution 2.2 Maximum Likelihood and Hypothesis Test

(a) Since $\log Y_1, \ldots, \log Y_8$ are independent random variables, the joint density $f_{\mu,\sigma^2}(x_1, \ldots, x_8)$ of $\log Y_1, \ldots, \log Y_8$ is given by product of the marginal densities of $\log Y_1, \ldots, \log Y_8$. We have

$$f_{\mu,\sigma^2}(x_1, \ldots, x_8) = \prod_{i=1}^8 \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\frac{(x_i - \mu)^2}{\sigma^2}\right\},$$

since $\log Y_1, \ldots, \log Y_8$ are Gaussian random variables with mean μ and variance σ^2.

(b) By taking the logarithm, we get

$$\begin{aligned}\log f_{\mu,\sigma^2}(x_1,\dots,x_8) &= \sum_{i=1}^{8} -\log\left(\sqrt{2\pi}\right) - \log(\sigma) - \frac{1}{2}\frac{(x_i-\mu)^2}{\sigma^2}\\ &= -8\log\left(\sqrt{2\pi}\right) - 8\log(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{8}(x_i-\mu)^2.\end{aligned}$$

(c) We have $\log f_{\mu,\sigma^2}(x_1,\dots,x_8) < -8\log(\sigma)$ for all $\mu \in \mathbb{R}$. Hence, independently of μ, $\log f_{\mu,\sigma^2}(x_1,\dots,x_8) \to -\infty$ if $\sigma^2 \to \infty$. Moreover, since for example $x_1 \neq x_2$, there exists a $c > 0$ with $\sum_{i=1}^{8}(x_i-\mu)^2 > c$ and thus $\log f_{\mu,\sigma^2}(x_1,\dots,x_8) < -8\log(\sigma) - \frac{c}{2\sigma^2}$ for all $\mu \in \mathbb{R}$. Since $\frac{c}{2\sigma^2}$ goes much faster to ∞ than $8\log(\sigma)$ goes to $-\infty$ if $\sigma^2 \to 0$, we have $\log f_{\mu,\sigma^2}(x_1,\dots,x_8) \to -\infty$ if $\sigma^2 \to 0$, independently of μ. Finally, if $\sigma^2 \in [c_1,c_2]$ for some $0 < c_1 < c_2$, we have $\log f_{\mu,\sigma^2}(x_1,\dots,x_8) < -\frac{1}{2c_2}\sum_{i=1}^{8}(x_i-\mu)^2$. Hence, independently of the value of σ^2 in the interval $[c_1,c_2]$, $\log f_{\mu,\sigma^2}(x_1,\dots,x_8) \to -\infty$ if $|\mu| \to \infty$. Since $\log f_{\mu,\sigma^2}(x_1,\dots,x_8)$ is continuous in μ and σ^2, we can conclude that it attains its global maximum somewhere in $\mathbb{R}\times\mathbb{R}_{>0}$. Thus $\widehat{\mu}$ and $\widehat{\sigma}^2$ as defined on the exercise sheet have to satisfy the first order conditions

$$\frac{\partial}{\partial\mu}\log f_{\mu,\sigma^2}(x_1,\dots,x_8)|_{(\mu,\sigma^2)=(\hat{\mu},\hat{\sigma}^2)} = 0 \qquad \text{and}$$
$$\frac{\partial}{\partial(\sigma^2)}\log f_{\mu,\sigma^2}(x_1,\dots,x_8)|_{(\mu,\sigma^2)=(\hat{\mu},\hat{\sigma}^2)} = 0.$$

We calculate

$$\frac{\partial}{\partial\mu}\log f_{\mu,\sigma^2}(x_1,\dots,x_8) = \frac{1}{\sigma^2}\sum_{i=1}^{8}(x_i-\mu),$$

which is equal to 0 if and only if $\mu = \frac{1}{8}\sum_{i=1}^{8}x_i$. Moreover, we have

$$\frac{\partial}{\partial(\sigma^2)}\log f_{\mu,\sigma^2}(x_1,\dots,x_8) = -\frac{8}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{8}(x_i-\mu)^2 = \frac{1}{2\sigma^2}\left[-8 + \frac{1}{\sigma^2}\sum_{i=1}^{8}(x_i-\mu)^2\right],$$

which is equal to 0 if and only if $\sigma^2 = \frac{1}{8}\sum_{i=1}^{8}(x_i-\mu)^2$. Since there is only tuple in $\mathbb{R}\times\mathbb{R}_{>0}$ that satisfies the first order conditions, we conclude that

$$\widehat{\mu} = \frac{1}{8}\sum_{i=1}^{8}x_i = 7 \qquad \text{and} \qquad \widehat{\sigma}^2 = \frac{1}{8}\sum_{i=1}^{8}(x_i-\widehat{\mu})^2 = \frac{1}{8}\sum_{i=1}^{8}(x_i-7)^2 = 7.$$

Note that the MLE $\widehat{\sigma}^2$ (considered as an estimator) is not unbiased. Indeed, if we replace $x_1,\dots,x_8$ by independent Gaussian random variables $X_1,\dots,X_8$ with expectation $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ and write $\widehat{\mu}$ for $\frac{1}{8}\sum_{i=1}^{8}X_i$, we can calculate

$$\mathbb{E}[\widehat{\sigma}^2] = \mathbb{E}[\widehat{\sigma}^2(X_1,\dots,X_8)] = \mathbb{E}\left[\frac{1}{8}\sum_{i=1}^{8}(X_i-\widehat{\mu})^2\right] = \frac{1}{8}\mathbb{E}\left[\sum_{i=1}^{8}(X_i^2 - 2X_i\widehat{\mu} + \widehat{\mu}^2)\right].$$

By noting that $\sum_{i=1}^{8}X_i = 8\widehat{\mu}$ and that $\mathbb{E}[X_1^2] = \dots = \mathbb{E}[X_8^2]$, we get

$$\mathbb{E}[\widehat{\sigma}^2] = \frac{1}{8}\mathbb{E}\left[\sum_{i=1}^{8}X_i^2 - 2\cdot 8\cdot\widehat{\mu}^2 + 8\widehat{\mu}^2\right] = \mathbb{E}[X_1^2] - \mathbb{E}[\widehat{\mu}^2] = \sigma^2 + \mathbb{E}[X_1]^2 - \mathrm{Var}(\widehat{\mu}) - \mathbb{E}[\widehat{\mu}]^2.$$

By inserting

$$\mathrm{Var}(\widehat{\mu}) = \mathrm{Var}\left(\frac{1}{8}\sum_{i=1}^{8} X_i\right) = \left(\frac{1}{8}\right)^2 \sum_{i=1}^{8} \mathrm{Var}(X_i) = \frac{1}{8}\sigma^2 \qquad \text{and}$$

$$\mathbb{E}[\widehat{\mu}]^2 = \mathbb{E}\left[\frac{1}{8}\sum_{i=1}^{8} X_i\right]^2 = \left(\frac{1}{8}\sum_{i=1}^{8} \mathbb{E}[X_i]\right)^2 = \mathbb{E}[X_1]^2,$$

we can conclude that

$$\mathbb{E}[\widehat{\sigma}^2] = \sigma^2 + \mathbb{E}[X_1]^2 - \frac{1}{8}\sigma^2 - \mathbb{E}[X_1]^2 = \frac{7}{8}\sigma^2 \neq \sigma^2,$$

i.e. $\widehat{\sigma}^2$ is not unbiased.

(d) Since our data is assumed to follow a Gaussian distribution and the variance is unknown, we perform a t-test. The test statistic is given by

$$T = T(\log Y_1, \ldots, \log Y_8) = \sqrt{8}\,\frac{\frac{1}{8}\sum_{i=1}^{8} \log Y_i - \mu}{\sqrt{S^2}},$$

where

$$S^2 = \frac{1}{7}\sum_{i=1}^{8}\left(\log Y_i - \frac{1}{8}\sum_{i=1}^{8}\log Y_i\right)^2.$$

Under H_0, T follows a Student-t distribution with 7 degrees of freedom. With the data given on the exercise sheet, the random variable S^2 attains the value

$$\frac{1}{7}\sum_{i=1}^{8}\left(x_i - \frac{1}{8}\sum_{i=1}^{8} x_i\right)^2 = \frac{1}{7}\sum_{i=1}^{8}(x_i - 7)^2 = 8.$$

Thus, for T we get the observation

$$T(x_1, \ldots, x_8) = \sqrt{8}\,\frac{\frac{1}{8}\sum_{i=1}^{8} x_i - \mu}{\sqrt{S^2}} = \sqrt{8}\,\frac{7-6}{\sqrt{8}} = 1,$$

where we use that $\mu = 6$ under H_0. Now the probability under H_0 to observe a T that is at least as extreme as the observation 1 we got above is

$$\mathbb{P}[|T| \geq 1] = \mathbb{P}[T \geq 1] + \mathbb{P}[T \leq -1] = 1 - \mathbb{P}[T < 1] + 1 - \mathbb{P}[T < 1] = 2 - 2\mathbb{P}[T < 1],$$

where we used the symmetry of the Student-t distribution around 0. The probability $\mathbb{P}[T < 1]$ is approximately 0.83, and the p-value is given by

$$\mathbb{P}[|T| \geq 1] = 2 - 2\mathbb{P}[T < 1] \approx 2 - 2 \cdot 0.83 = 0.34.$$

This p-value is fairly high, and we conclude that we can not reject the null hypothesis, for example, at significance level of 5% or 1%.

Solution 2.3 $\mathcal{X}^2$-Distribution

(a) Let $r \in \mathbb{R}$. The moment generating function M_{X_k} of X_k can be calculated as follows

$$\begin{aligned} M_{X_k}(r) = \mathbb{E}\left[\exp\{rX_k\}\right] &= \int_{\mathbb{R}} \exp\{rx\} f_{X_k}(x)\,dx \\ &= \int_0^{\infty} \exp\{rx\}\frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1}\exp\{-x/2\}\,dx \\ &= \int_0^{\infty} \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1}\exp\{-x(1/2-r)\}\,dx. \end{aligned}$$

This integral (and consequently the moment generating function) exists if and only if $r < 1/2$. Let $r < 1/2$. Then, we use the substitution

$$u = x(1/2 - r), \quad dx = \frac{1}{1/2 - r} du.$$

We get

$$\begin{aligned} M_{X_k}(r) &= \int_0^\infty \frac{1}{2^{k/2}\Gamma(k/2)} u^{k/2-1} \left(\frac{1}{1/2-r}\right)^{k/2-1} \exp\{-u\} \frac{1}{1/2-r}\, du \\ &= \frac{1}{2^{k/2}} \left(\frac{1}{1/2-r}\right)^{k/2} \frac{1}{\Gamma(k/2)} \int_0^\infty u^{k/2-1} \exp\{-u\}\, du \\ &= \left(\frac{1}{1-2r}\right)^{k/2}, \end{aligned}$$

where we used the definition of the gamma function

$$\Gamma(z) = \int_0^\infty u^{z-1} \exp\{-u\}\, du, \quad \text{for } z \in \mathbb{R}.$$

(b) For all $r < 1/2$ the moment generating function M_{Z^2} of Z^2 is given by

$$\begin{aligned} M_{Z^2}(r) = \mathbb{E}\left[\exp\left\{rZ^2\right\}\right] &= \int_{-\infty}^\infty \exp\{rx^2\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx \\ &= (1-2r)^{-1/2} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}(1-2r)^{-1/2}} \exp\left\{-\frac{x^2}{2(1-2r)^{-1}}\right\} dx \\ &= \left(\frac{1}{1-2r}\right)^{1/2} \\ &= M_{X_1}(r), \end{aligned}$$

where the second to last equality holds true since we integrate the density of a normal distribution with mean 0 and variance $(1-2r)^{-1}$. Since the moment generating function uniquely determines the distribution, we conclude that $Z^2 \overset{(d)}{=} X_1$.

(c) Using that $Z_1, \dots, Z_k$ are i.i.d., the moment generating function M_Y of $Y \overset{\text{def}}{=} \sum_{i=1}^k Z_i^2$ is given by

$$M_Y(r) = \mathbb{E}\left[\exp\left\{r\sum_{i=1}^k Z_i^2\right\}\right] = \prod_{i=1}^k \mathbb{E}\left[\exp\left\{rZ_i^2\right\}\right] = \Big(M_{Z_1^2}(r)\Big)^k = \left(\frac{1}{1-2r}\right)^{k/2} = M_{X_k}(r),$$

for all $r < 1/2$. Since the moment generating function uniquely determines the distribution, we conclude that $\sum_{i=1}^k Z_i^2 \overset{(d)}{=} X_k$.

Solution 2.4 Variance Decomposition

By definition of the random variable X, the second moments exist. Hence, we have

$$\mathbb{E}[\mathrm{Var}(X|\mathcal{G})] = \mathbb{E}\left[\mathbb{E}[X^2|\mathcal{G}] - (\mathbb{E}[X|\mathcal{G}])^2\right] = \mathbb{E}[X^2] - \mathbb{E}\left[(\mathbb{E}[X|\mathcal{G}])^2\right]$$

and

$$\mathrm{Var}(\mathbb{E}[X|\mathcal{G}]) = \mathbb{E}\left[(\mathbb{E}[X|\mathcal{G}])^2\right] - \mathbb{E}\left[\mathbb{E}[X|\mathcal{G}]\right]^2 = \mathbb{E}\left[(\mathbb{E}[X|\mathcal{G}])^2\right] - \mathbb{E}[X]^2.$$

Combining these two results, we get

$$\mathbb{E}[\mathrm{Var}(X|\mathcal{G})] + \mathrm{Var}(\mathbb{E}[X|\mathcal{G}]) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathrm{Var}(X).$$