Non-Life Insurance: Mathematics and Statistics

Solution sheet 3

Solution 3.1 No-Claims Bonus

(a) We define the following events:

$$
\begin{aligned}
A &= \{\text{“no claims in the last six years”}\},\\
B &= \{\text{“no claims in the last three years but at least one claim in the last six years”}\},\\
C &= \{\text{“at least one claim in the last three years”}\}.
\end{aligned}
$$

Note that since the events A, B and C are disjoint and cover all possible outcomes, we have

$$\mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C] = 1,$$

i.e. it is sufficient to calculate two out of the three probabilities. Since the calculation of $\mathbb{P}[B]$ is slightly more involved, we will look at $\mathbb{P}[A]$ and $\mathbb{P}[C]$. Let $N_1, \ldots, N_6$ be the number of claims of the last six years of our considered car driver, where N_6 corresponds to the most recent year. By assumption, $N_1, \ldots, N_6$ are independent Poisson random variables with frequency parameter $\lambda = 0.2$. Therefore, we can calculate

$$\mathbb{P}[A] = \mathbb{P}\left[N_1 = 0, \ldots, N_6 = 0\right] = \prod_{i=1}^{6} \mathbb{P}\left[N_i = 0\right] = \prod_{i=1}^{6} \exp\{-\lambda\} = \exp\{-6\lambda\} = \exp\{-1.2\}$$

and, similarly,

$$\mathbb{P}[C] = 1 - \mathbb{P}[C^c] = 1 - \mathbb{P}\left[N_4 = 0, N_5 = 0, N_6 = 0\right] = 1 - \exp\{-3\lambda\} = 1 - \exp\{-0.6\}.$$

For the event B we get

$$\mathbb{P}[B] = 1 - \mathbb{P}[A] - \mathbb{P}[C] = 1 - \exp\{-1.2\} - (1 - \exp\{-0.6\}) = \exp\{-0.6\} - \exp\{-1.2\}.$$

Thus, the expected proportion q of the premium that is still paid after the grant of the no-claims bonus is given by

$$
\begin{aligned}
q &= 0.8 \cdot \mathbb{P}[A] + 0.9 \cdot \mathbb{P}[B] + 1 \cdot \mathbb{P}[C]\\
&= 0.8 \cdot \exp\{-1.2\} + 0.9 \cdot (\exp\{-0.6\} - \exp\{-1.2\}) + 1 - \exp\{-0.6\}\\
&\approx 0.915.
\end{aligned}
$$

If s denotes the surcharge on the premium, then it has to satisfy the equation

$$q(1 + s) \cdot \text{premium} = \text{premium},$$

which leads to

$$s = \frac{1}{q} - 1.$$

We conclude that the surcharge on the premium is given by approximately 9.3%.

(b) We use the same notation as in (a). Since this time the calculation of $\mathbb{P}[B]$ is considerably more involved, we again look at $\mathbb{P}[A]$ and $\mathbb{P}[C]$. By assumption, conditionally given Θ, $N_1, \ldots, N_6$ are independent Poisson random variables with frequency parameter $\Theta\lambda$, where $\lambda = 0.2$. Therefore, we can calculate

$$\begin{aligned}
\mathbb{P}[A] &= \mathbb{P}\left[N_1 = 0, \ldots, N_6 = 0\right] \\
&= \mathbb{E}\left[\mathbb{P}\left[N_1 = 0, \ldots, N_6 = 0 | \Theta\right]\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{6} \mathbb{P}\left[N_i = 0 | \Theta\right]\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{6} \exp\{-\Theta\lambda\}\right] \\
&= \mathbb{E}\left[\exp\{-6\Theta\lambda\}\right] \\
&= M_\Theta(-6\lambda),
\end{aligned}$$

where M_Θ denotes the moment generating function of Θ. Since $\Theta \sim \Gamma(1,1)$, M_Θ is given by

$$M_\Theta(r) = \frac{1}{1-r},$$

for all $r < 1$, which leads to

$$\mathbb{P}[A] = \frac{1}{1+6\lambda} = \frac{1}{2.2}.$$

Similarly, we get

$$\mathbb{P}[C] = 1 - \mathbb{P}[C^c] = 1 - \mathbb{P}\left[N_4 = 0, N_5 = 0, N_6 = 0\right] = 1 - \frac{1}{1+3\lambda} = 1 - \frac{1}{1.6} = \frac{0.6}{1.6}.$$

For the event B we get

$$\mathbb{P}[B] = 1 - \mathbb{P}[A] - \mathbb{P}[C] = 1 - \frac{1}{2.2} - \frac{0.6}{1.6} = \frac{1}{1.6} - \frac{1}{2.2}.$$

Thus, the expected proportion q of the premium that is still paid after the grant of the no-claims bonus is given by

$$\begin{aligned}
q &= 0.8 \cdot \mathbb{P}[A] + 0.9 \cdot \mathbb{P}[B] + 1 \cdot \mathbb{P}[C] \\
&= 0.8 \cdot \frac{1}{2.2} + 0.9 \cdot \left(\frac{1}{1.6} - \frac{1}{2.2}\right) + \frac{0.6}{1.6} \\
&\approx 0.892.
\end{aligned}$$

We conclude that the surcharge s on the premium is given by

$$s = \frac{1}{q} - 1 \approx 12.1\%,$$

which is considerably bigger than in (a). The reason is that in (b) we introduce dependence between the claim counts of the individual years of the considered car driver. This increases the probability of having no claims in the last six years, and decreases the expected proportion q of the premium that is still paid after the grant of the no-claims bonus.

Solution 3.2 Claims Count Distribution

The sample mean and the sample variance of the observed numbers of claims $N_1, \ldots, N_{10}$ are given by

$$\widehat{\mu} \stackrel{\text{def}}{=} \frac{1}{10} \sum_{t=1}^{10} N_t = 21.3 \qquad \text{and} \qquad \widehat{\sigma}^2 \stackrel{\text{def}}{=} \frac{1}{9} \sum_{t=1}^{10} (N_t - \widehat{\mu})^2 \approx 109.1.$$

We have

$$\widehat{\sigma}^2 \approx 5\widehat{\mu},$$

which suggests $\mathrm{Var}(N_1) > \mathbb{E}[N_1]$. In such a case we would choose a negative binomial distribution for modeling the number of claims, as it is the only distribution among the three distributions mentioned which allows the variance to exceed the expectation.

Solution 3.3 Central Limit Theorem

Let σ^2 be the variance of the claim sizes and $x > 0$. We have

$$\begin{aligned}
\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^n Y_i - \mu\right| < \frac{x}{\sqrt{n}}\right] &= \mathbb{P}\left[\frac{1}{n}\sum_{i=1}^n Y_i - \mu < \frac{x}{\sqrt{n}}\right] - \mathbb{P}\left[\frac{1}{n}\sum_{i=1}^n Y_i - \mu \le -\frac{x}{\sqrt{n}}\right] \\
&= \mathbb{P}\left[\sqrt{n}\frac{\frac{1}{n}\sum_{i=1}^n Y_i - \mu}{\sigma} < \frac{x}{\sigma}\right] - \mathbb{P}\left[\sqrt{n}\frac{\frac{1}{n}\sum_{i=1}^n Y_i - \mu}{\sigma} \le -\frac{x}{\sigma}\right] \\
&= \mathbb{P}\left[Z_n < \frac{x}{\sigma}\right] - \mathbb{P}\left[Z_n \le -\frac{x}{\sigma}\right],
\end{aligned}$$

where

$$Z_n = \sqrt{n}\frac{\frac{1}{n}\sum_{i=1}^n Y_i - \mu}{\sigma}.$$

According to the Central Limit Theorem, Z_n converges in distribution to a standard Gaussian random variable. Hence, if we write Φ for the distribution function of a standard Gaussian random variable, we have the approximation

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^n Y_i - \mu\right| < \frac{x}{\sqrt{n}}\right] \approx \Phi\left(\frac{x}{\sigma}\right) - \Phi\left(-\frac{x}{\sigma}\right).$$

On the one hand, as we are interested in a probabilty of at least 95%, we have to choose $x > 0$ such that

$$\Phi\left(\frac{x}{\sigma}\right) - \Phi\left(-\frac{x}{\sigma}\right) = 0.95.$$

Using $\Phi(-\frac{x}{\sigma}) = 1 - \Phi(\frac{x}{\sigma})$ and $\Phi^{-1}(0.975) = 1.96$, this implies that

$$\frac{x}{\sigma} = 1.96.$$

It follows that

$$x = 1.96 \cdot \sigma = 1.96 \cdot \mathrm{Vco}(Y_1) \cdot \mu = 1.96 \cdot 4 \cdot \mu. \tag{1}$$

On the other hand, as we want the deviation of the empirical mean from μ to be less than 1%, we set

$$\frac{x}{\sqrt{n}} = 0.01 \cdot \mu,$$

which implies

$$n = \frac{x^2}{0.01^2 \cdot \mu^2}. \tag{2}$$

Combining (1) and (2), we conclude that

$$n = \frac{(1.96 \cdot 4 \cdot \mu)^2}{0.01^2 \cdot \mu^2} = 1.96^2 \cdot 4^2 \cdot 10'000 = 614'656.$$

Solution 3.4 Compound Binomial Distribution

(a) Let $\tilde{S} \sim \text{CompBinom}(\tilde{v}, \tilde{p}, \tilde{G})$ with the random variable $\tilde{Y}_1$ having distribution function $\tilde{G}$ and moment generating function $M_{\tilde{Y}_1}$. Then, by Proposition 2.6 of the lecture notes, the moment generating function $M_{\tilde{S}}$ of $\tilde{S}$ is given by

$$M_{\tilde{S}}(r) = \left(\tilde{p} M_{\tilde{Y}_1}(r) + 1 - \tilde{p}\right)^{\tilde{v}},$$

for all $r \in \mathbb{R}$ for which $M_{\tilde{Y}_1}$ is defined. We calculate the moment generating function of S_{lc} and show that it is exactly of the form given above. Since $S_{\text{lc}} \geq 0$ almost surely, its moment generating function is defined at least for all $r < 0$. Thus, for $r < 0$, we have

$$\begin{aligned} M_{S_{\text{lc}}}(r) &= \mathbb{E}\left[\exp\left\{r \sum_{i=1}^{N} Y_i \, 1_{\{Y_i > M\}}\right\}\right] \\ &= \mathbb{E}\left[\prod_{i=1}^{N} \exp\left\{rY_i \, 1_{\{Y_i > M\}}\right\}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{N} \exp\left\{rY_i \, 1_{\{Y_i > M\}}\right\} \middle| N\right]\right] \\ &= \mathbb{E}\left[\prod_{i=1}^{N} \mathbb{E}\left[\exp\left\{rY_i \, 1_{\{Y_i > M\}}\right\}\right]\right], \end{aligned}$$

where in the third equality we used the tower property of conditional expectation and in the fourth equality the independence between N and Y_i. For the inner expectation we get

$$\begin{aligned} \mathbb{E}\left[\exp\left\{rY_i \, 1_{\{Y_i > M\}}\right\}\right] &= \mathbb{E}\left[\exp\left\{rY_i\right\} \cdot 1_{\{Y_i > M\}} + 1_{\{Y_i \leq M\}}\right] \\ &= \mathbb{E}\left[\exp\left\{rY_i\right\} | Y_i > M\right] \mathbb{P}[Y_i > M] + \mathbb{P}[Y_i \leq M] \\ &= \mathbb{E}\left[\exp\left\{rY_i\right\} | Y_i > M\right] [1 - G(M)] + G(M). \end{aligned}$$

First, note that the distribution function of the random variable $Y_i | Y_i > M$ is G_{lc}. Moreover, since $Y_i | Y_i > M$ is greater than 0 almost surely, its moment generating function $M_{Y_1 | Y_1 > M}$ is defined for all $r < 0$. Thus, we can write

$$\mathbb{E}\left[\exp\left\{rY_i \, 1_{\{Y_i > M\}}\right\}\right] = M_{Y_1 | Y_1 > M}(r)[1 - G(M)] + G(M).$$

Hence, we get

$$\begin{aligned} M_{S_{\text{lc}}}(r) &= \mathbb{E}\left[\prod_{i=1}^{N} \left(M_{Y_1 | Y_1 > M}(r)[1 - G(M)] + G(M)\right)\right] \\ &= \mathbb{E}\left[\left(M_{Y_1 | Y_1 > M}(r)[1 - G(M)] + G(M)\right)^N\right] \\ &= \mathbb{E}\left[\exp\left\{N \log\left(M_{Y_1 | Y_1 > M}(r)[1 - G(M)] + G(M)\right)\right\}\right] \\ &= M_N(\rho), \end{aligned}$$

where M_N is the moment generating function of N and

$$\rho = \log\left(M_{Y_1 | Y_1 > M}(r)[1 - G(M)] + G(M)\right).$$

Since we have $N \sim \text{Binom}(v, p)$, $M_N(r)$ is given by

$$M_N(r) = (p \exp\{r\} + 1 - p)^v.$$

Therefore, we get

$$\begin{aligned} M_{S_{\mathrm{lc}}}(r) &= \left[p\left(M_{Y_1|Y_1>M}(r)[1-G(M)] + G(M)\right) + 1 - p\right]^v \\ &= \left(p[1-G(M)]M_{Y_1|Y_1>M}(r) + 1 - p[1-G(M)]\right)^v. \end{aligned}$$

Applying Lemma 1.3 of the lecture notes, we conclude that $S_{\mathrm{lc}} \sim \mathrm{CompBinom}(\tilde{v}, \tilde{p}, \tilde{G})$ with $\tilde{v} = v$, $\tilde{p} = p[1-G(M)]$ and $\tilde{G} = G_{\mathrm{lc}}$.

(b) In (a) we showed that the number of claims of the compound distribution S_{lc} has a binomial distribution with parameters v and $1 - G(M)$. In particular, there is a positive probability that we have v claims with $Y_i > M$. Now suppose that $S_{\mathrm{sc}} > 0$. Then, we know that there is an $i \in \{1, \ldots, N\}$ with $Y_i \leq M$. In particular, this claim cannot be part of S_{lc} and there is zero probability that we have v claims with $Y_i > M$. This explains why S_{sc} and S_{lc} cannot be independent. However, note that with the Poisson distribution as claims count distribution such a split in small and large claims leads to independent compound distributions, see Theorem 2.14 of the lecture notes.