Non-Life Insurance: Mathematics and Statistics

Solution sheet 4

Solution 4.1 Poisson Model and Negative-Binomial Model

(a) In the Poisson model we assume that $N_1, \ldots, N_{10}$ are independent with $N_t \sim \mathrm{Poi}(\lambda v_t)$ for all $t \in \{1, \ldots, 10\}$. We use Estimator 2.32 of the lecture notes to estimate the claims frequency parameter λ by

$$\widehat{\lambda}_{10}^{\mathrm{MLE}} = \frac{\sum_{t=1}^{10} N_t}{\sum_{t=1}^{10} v_t} = \frac{10\text{'}224}{100\text{'}000} \approx 10.22\%.$$

Let $t \in \{1, \ldots, 10\}$. We have

$$\mathbb{E}\left[\frac{N_t}{v_t}\right] = \frac{\mathbb{E}[N_t]}{v_t} = \frac{\lambda v_t}{v_t} = \lambda \qquad \text{and} \qquad \mathrm{Var}\left(\frac{N_t}{v_t}\right) = \frac{\mathrm{Var}(N_t)}{v_t^2} = \frac{\lambda v_t}{v_t^2} = \frac{\lambda}{v_t}.$$

Note that the random variable $N_t \sim \mathrm{Poi}(\lambda v_t)$ can be understood as

$$N_t \overset{(\mathrm{d})}{=} \sum_{i=1}^{v_t} \widetilde{N}_i,$$

where $\widetilde{N}_1, \ldots, \widetilde{N}_{v_t}$ are independent random variables that all follow a $\mathrm{Poi}(\lambda)$-distribution. Thus, we can use the Central Limit Theorem to get

$$\frac{N_t/v_t - \mathbb{E}\left[N_t/v_t\right]}{\sqrt{\mathrm{Var}\left(N_t/v_t\right)}} = \frac{N_t/v_t - \lambda}{\sqrt{\lambda/v_t}} \longrightarrow Z,$$

as $v_t \to \infty$, where Z is a random variable following a standard normal distribution. This leads to the approximation

$$\mathbb{P}\left[\lambda - \sqrt{\lambda/v_t} \leq N_t/v_t \leq \lambda + \sqrt{\lambda/v_t}\right] = \mathbb{P}\left[-1 \leq \frac{N_t/v_t - \lambda}{\sqrt{\lambda/v_t}} \leq 1\right] \approx \mathbb{P}(-1 \leq Z \leq 1) \approx 0.7,$$

i.e. with a probability of roughly 70%, N_t/v_t lies in the interval $[\lambda - \sqrt{\lambda/v_t}, \lambda + \sqrt{\lambda/v_t}]$. Since λ is unknown, we replace it by the estimator $\widehat{\lambda}_{10}^{\mathrm{MLE}}$ to get the approximate interval

$$\left[\widehat{\lambda}_{10}^{\mathrm{MLE}} - \sqrt{\widehat{\lambda}_{10}^{\mathrm{MLE}}/v_t}, \widehat{\lambda}_{10}^{\mathrm{MLE}} + \sqrt{\widehat{\lambda}_{10}^{\mathrm{MLE}}/v_t}\right] \approx [9.90\%, 10.54\%],$$

which should contain roughly 70% of the observed claims frequencies N_t/v_t. We have the following observations of the claims frequencies:

t	1	2	3	4	5	6	7	8	9	10
N_t/v_t	10%	9.97%	9.85%	9.89%	10.56%	10.70%	9.94%	9.86%	10.93%	10.54%

Table 1: Observed claims frequencies N_t/v_t.

We observe that instead of the expected, roughly seven observations, only four observations lie in the estimated interval. We conclude that the assumption of having Poisson distributions might not be reasonable.

(b) By equation (2.8) of the lecture notes, the test statistic $\widehat{\chi}^*$ is given by

$$\widehat{\chi}^* = \sum_{t=1}^{10} v_t \frac{\left(N_t/v_t - \widehat{\lambda}_{10}^{\mathrm{MLE}}\right)^2}{\widehat{\lambda}_{10}^{\mathrm{MLE}}}$$

and is approximately χ^2-distributed with $10 - 1 = 9$ degrees of freedom. By inserting the numbers and $\widehat{\lambda}_{10}^{\mathrm{MLE}}$ calculated in (a), we get

$$\widehat{\chi}^* \approx 14.84.$$

The probability that a random variable with a χ^2-distribution with 9 degrees of freedom is greater than 14.84 is approximately equal to 9.55%. Hence we can reject the null hypothesis of having Poisson distributions only at significance levels that are higher than 9.55%. In particular, we can not reject the null hypothesis at the significance level of 5%.

(c) In the negative-binomial model we assume that $N_1, \ldots, N_{10}$ are independent with, conditionally given Θ_t, $N_t \sim \mathrm{Poi}(\Theta_t \lambda v_t)$ for all $t \in \{1, \ldots, 10\}$, where $\Theta_1, \ldots, \Theta_{10} \overset{\text{i.i.d.}}{\sim} \Gamma(\gamma, \gamma)$ for some $\gamma > 0$. We use Estimator 2.28 of the lecture notes to estimate the claims frequency parameter λ by

$$\widehat{\lambda}_{10}^{\mathrm{NB}} = \frac{\sum_{t=1}^{10} N_t}{\sum_{t=1}^{10} v_t} = \frac{10\text{'}224}{100\text{'}000} \approx 10.22\%.$$

As in equation (2.7) of the lecture notes, we define

$$\widehat{V}_{10}^2 = \frac{1}{9} \sum_{t=1}^{10} v_t \left(\frac{N_t}{v_t} - \widehat{\lambda}_{10}^{\mathrm{NB}} \right)^2 \approx 16.9\%.$$

Let $v = v_1 = \cdots = v_{10} = 10\text{'}000$. Now we can use Estimator 2.30 of the lecture notes to estimate the dispersion parameter γ by

$$\widehat{\gamma}_{10}^{\mathrm{NB}} = \frac{\left(\widehat{\lambda}_{10}^{\mathrm{NB}}\right)^2}{\widehat{V}_{10}^2 - \widehat{\lambda}_{10}^{\mathrm{NB}}} \frac{1}{9} \left(\sum_{t=1}^{10} v_t - \frac{\sum_{t=1}^{10} v_t^2}{\sum_{t=1}^{10} v_t} \right) = \frac{\left(\widehat{\lambda}_{10}^{\mathrm{NB}}\right)^2}{\widehat{V}_{10}^2 - \widehat{\lambda}_{10}^{\mathrm{NB}}} \frac{\left(10v - \frac{10v^2}{10v}\right)}{9} = \frac{\left(\widehat{\lambda}_{10}^{\mathrm{NB}}\right)^2 v}{\widehat{V}_{10}^2 - \widehat{\lambda}_{10}^{\mathrm{NB}}} \approx 1576.15.$$

For all $t \in \{1, \ldots, 10\}$ we have

$$\mathbb{E}\left[\frac{N_t}{v_t}\right] = \frac{\mathbb{E}[N_t]}{v_t} = \frac{\mathbb{E}[\mathbb{E}[N_t|\Theta_t]]}{v_t} = \frac{\mathbb{E}[\Theta_t \lambda v_t]}{v_t} = \frac{\lambda v_t}{v_t} = \lambda,$$

since $\mathbb{E}[\Theta_t] = 1$, and

$$\mathrm{Var}\left(\frac{N_t}{v_t}\right) = \frac{\mathbb{E}[\mathrm{Var}(N_t|\Theta_t)] + \mathrm{Var}(\mathbb{E}[N_t|\Theta_t])}{v_t^2} = \frac{\mathbb{E}[\Theta_t \lambda v_t] + \mathrm{Var}(\Theta_t \lambda v_t)}{v_t^2} = \frac{\lambda + \frac{\lambda^2 v_t}{\gamma}}{v_t},$$

since $\mathrm{Var}(\Theta_t) = 1/\gamma$. Similarly as in the Poisson case in part (a), we get the estimated interval

$$\left[\widehat{\lambda}_{10}^{\mathrm{NB}} - \sqrt{\frac{\widehat{\lambda}_{10}^{\mathrm{NB}} + \left(\widehat{\lambda}_{10}^{\mathrm{NB}}\right)^2 v_t / \widehat{\gamma}_{10}^{\mathrm{NB}}}{v_t}}, \widehat{\lambda}_{10}^{\mathrm{NB}} + \sqrt{\frac{\widehat{\lambda}_{10}^{\mathrm{NB}} + \left(\widehat{\lambda}_{10}^{\mathrm{NB}}\right)^2 v_t / \widehat{\gamma}_{10}^{\mathrm{NB}}}{v_t}} \right] \approx [9.81\%, 10.63\%],$$

which should contain roughly 70% of the observed claims frequencies N_t/v_t. Looking at the observations given in Table 1 above, we see that eight of them lie in the estimated interval, which is clearly better than in the Poisson case in part (a). In conclusion, the negative-binomial model seems more reasonable than the Poisson model.

Solution 4.2 χ^2-Goodness-of-Fit-Analysis (R Exercise)

The R Code used in this exercise is provided below.

(a) (i) In Figure 1 (left) we can see that the n MLEs of λ approximately have a Gaussian distribution with mean equal to the true value of $\lambda = 10\%$. On the one hand, this is due to the fact that (under regularity assumptions) the MLE is consistent and asymptotically Gaussian distributed (as $T \to \infty$). For more details we refer to Chapter 6 of the textbook "Theory of Point Estimation" by E.L. Lehmann and G. Casella (2nd edition, 1998). On the other hand, in the Poisson case we directly have an approximate Gaussian distribution of the MLE, independently of the value of T, provided that the volume v is large enough, see also the solution to Exercise 4.1.

(ii) From the QQ plot, see Figure 1 (right), we deduce that the test statistic indeed has approximately a χ^2-distribution with $T - 1 = 9$ degrees of freedom. We only observe slightly heavier tails in the observations, compared to a χ^2-distribution with $T - 1 = 9$ degrees of freedom. By increasing the values for n and v, we get even closer to a χ^2-distribution with $T - 1 = 9$ degrees of freedom.

(iii) We observe that we wrongly reject the null hypothesis H_0 of having a Poisson distribution as claims count distribution in 5.16% of the cases. This corresponds almost perfectly to the chosen significance level (indicating the probability of rejecting H_0 even though it is true) of 5%.

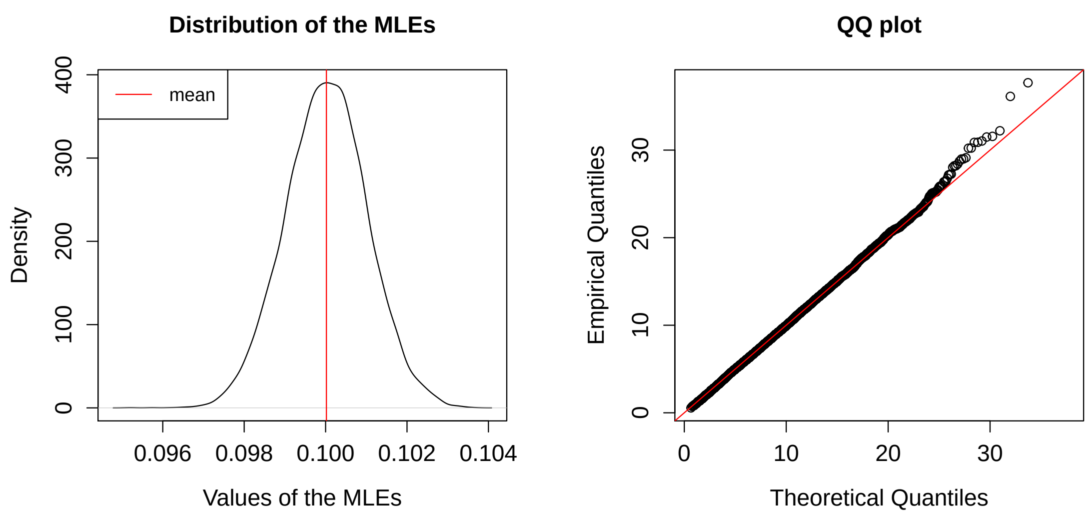

Figure 1: Plot of the distribution of the MLEs (left). QQ plot of the theoretical quantiles of a χ^2-distribution with $T - 1 = 9$ degrees of freedom against the empirical quantiles of the values of the test statistic (right).

(b) (i) We observe the following results:

	$\gamma = 100$	$\gamma =$1'000	$\gamma =$10'000
Percentage with which we reject H_0	99.78	48.38	7.96

Table 2: Percentage with which we reject H_0 for different values of γ.

(ii) We see that in case of a negative binomial distribution with a comparably small parameter ($\gamma = 100$) for the latent gamma distribution we are almost always able to reject the null hypothesis H_0 of having a Poisson distribution as claims count distribution. The bigger γ, the less we are able to reject H_0. This is because for very large values of γ, the corresponding gamma distribution does not vary a lot, i.e. is almost constantly equal to 1. Thus, for increasing γ, we move back to the Poisson model and, consequently, the χ^2-goodness-of-fit test does not detect the latent variable anymore.

```
### Exercise 2a)

### Define the function that generates the data and applies the
### chi-squared goodness-of-fit test in order to test the
### Poisson assumption
chi.squared.test.1 <- function(seed1, n, t, lambda, v, alpha){

  ### Generate the claims counts
  set.seed(seed1)
  claims.counts <- array(rpois(n*t,lambda*v), dim=c(t,n))

  ### Distribution of the MLEs of lambda
  lambda_MLE <- colSums(claims.counts)/(t*v)
  plot(density(lambda_MLE), main="Distribution of the MLEs",
      xlab="Values of the MLEs", cex.lab=1.25, cex.main=1.25,
      cex.axis=1.25)
  abline(v=mean(lambda_MLE), col="red")
  legend("topleft", lty=1, col="red", legend="mean")
  print("See plot for the distribution of the MLEs")

  ### Distribution of the test statistic
  lambda_MLE_array <- array(rep(lambda_MLE,each=t), dim=c(t,n))
  test.statistic <- colSums(v*(claims.counts/v-lambda_MLE_array
      )^2/lambda_MLE_array)
  theoretical.quantiles <- qchisq(p=(1:n)/(n+1), df=t-1)
  empirical.quantiles <- test.statistic[order(test.statistic)]
  lim <- c(min(theoretical.quantiles,empirical.quantiles), max(
      theoretical.quantiles,empirical.quantiles))
  plot(theoretical.quantiles, empirical.quantiles, xlim=lim,
      ylim=lim, xlab="Theoretical Quantiles", ylab="Empirical
      Quantiles", main="QQ plot", cex.lab=1.25, cex.main=1.25,
      cex.axis=1.25)
  abline(a=0, b=1, col="red")
  print("See the QQ plot for a comparison between the empirical
       quantiles of the test statistic and the theoretical
      quantiles of a chi-squared distribution with t-1 degrees
      of freedom")

  ### Result of the hypothesis test
  sum(test.statistic > qchisq(p=1-alpha, df=t-1))/n
}

### Apply the function with the desired parameters
chi.squared.test.1(seed1=100, n=10000, t=10, lambda=0.1, v
    =10000, alpha=0.05)
```

```r



### Exercise 2b)

### Define the function that generates the data and applies the
### chi-squared goodness-of-fit test in order to test the
### Poisson assumption
chi.squared.test.2 <- function(seed1, n, t, lambda, v, alpha, gamma){

  ### Generate the claims counts
  set.seed(seed1)
  claims.counts <- array(rnbinom(n*t, size = gamma, mu=lambda*v), dim=c(t,n))

  ### Calculate the MLEs
  lambda_MLE <- colSums(claims.counts)/(t*v)

  ### Calculate the test statistic
  lambda_MLE_array <- array(rep(lambda_MLE,each=t), dim=c(t,n))
  test.statistic <- colSums(v*(claims.counts/v-lambda_MLE_array)^2/lambda_MLE_array)

  ### Result of the hypothesis test
  sum(test.statistic > qchisq(p=1-alpha, df=t-1))/n
}

### Apply the function with the desired parameters
chi.squared.test.2(seed1=100, n=10000, t=10, lambda=0.1, v=10000, alpha=0.05, gamma=100)
chi.squared.test.2(seed1=100, n=10000, t=10, lambda=0.1, v=10000, alpha=0.05, gamma=1000)
chi.squared.test.2(seed1=100, n=10000, t=10, lambda=0.1, v=10000, alpha=0.05, gamma=10000)
```

Solution 4.3 Compound Poisson Distribution

(a) Since $S \sim \text{CompPoi}(\lambda v, G)$, we can write S as

$$S = \sum_{i=1}^{N} Y_i,$$

where $N \sim \text{Poi}(\lambda v)$, $Y_1, Y_2, \ldots$ are i.i.d. with distribution function G and N and $Y_1, Y_2, \ldots$ are independent. Now we can define S_{sc}, S_{mc} and S_{lc} as

$$S_{\text{sc}} = \sum_{i=1}^{N} Y_i 1_{\{Y_i \leq 1'000\}}, \quad S_{\text{mc}} = \sum_{i=1}^{N} Y_i 1_{\{1'000 < Y_i \leq 1'000'000\}} \quad \text{and} \quad S_{\text{lc}} = \sum_{i=1}^{N} Y_i 1_{\{Y_i > 1'000'000\}}.$$

(b) Note that according to Table 2 given on the exercise sheet, we have

$$\begin{aligned}
\mathbb{P}[Y_1 \leq 1\text{'}000] &= \mathbb{P}[Y = 100] + \mathbb{P}[Y = 300] + \mathbb{P}[Y = 500] = \frac{3}{20} + \frac{4}{20} + \frac{3}{20} = \frac{1}{2}, \\
\mathbb{P}[1\text{'}000 < Y_1 \leq 1\text{'}000\text{'}000] &= \mathbb{P}[Y = 6\text{'}000] + \mathbb{P}[Y = 100\text{'}000] + \mathbb{P}[Y = 500\text{'}000] \\
&= \frac{2}{15} + \frac{2}{15} + \frac{1}{15} \\
&= \frac{1}{3} \quad \text{and} \\
\mathbb{P}[Y_1 > 1\text{'}000\text{'}000] &= 1 - \mathbb{P}[Y_1 \leq 1\text{'}000\text{'}000] = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.
\end{aligned}$$

Thus, using Theorem 2.14 of the lecture notes (disjoint decomposition of compound Poisson distributions), we get

$$S_{\text{sc}} \sim \text{CompPoi}\left(\frac{\lambda v}{2}, G_{\text{sc}}\right), \quad S_{\text{mc}} \sim \text{CompPoi}\left(\frac{\lambda v}{3}, G_{\text{mc}}\right) \quad \text{and} \quad S_{\text{lc}} \sim \text{CompPoi}\left(\frac{\lambda v}{6}, G_{\text{lc}}\right),$$

where

$$\begin{aligned}
G_{\text{sc}}(y) &= \mathbb{P}[Y_1 \leq y | Y_1 \leq 1\text{'}000], \\
G_{\text{mc}}(y) &= \mathbb{P}[Y_1 \leq y | 1\text{'}000 < Y_1 \leq 1\text{'}000\text{'}000] \quad \text{and} \\
G_{\text{lc}}(y) &= \mathbb{P}[Y_1 \leq y | Y_1 > 1\text{'}000\text{'}000],
\end{aligned}$$

for all $y \in \mathbb{R}$. In particular, for a random variable Y_{sc} having distribution function G_{sc}, we have

$$\begin{aligned}
\mathbb{P}[Y_{\text{sc}} = 100] &= \frac{\mathbb{P}[Y = 100]}{\mathbb{P}[Y_1 \leq 1\text{'}000]} = \frac{3/20}{1/2} = \frac{3}{10}, \\
\mathbb{P}[Y_{\text{sc}} = 300] &= \frac{\mathbb{P}[Y = 300]}{\mathbb{P}[Y_1 \leq 1\text{'}000]} = \frac{4/20}{1/2} = \frac{4}{10} \quad \text{and} \\
\mathbb{P}[Y_{\text{sc}} = 500] &= \frac{\mathbb{P}[Y = 500]}{\mathbb{P}[Y_1 \leq 1\text{'}000]} = \frac{3/20}{1/2} = \frac{3}{10}.
\end{aligned}$$

Analogously, for random variables Y_{mc} and Y_{lc} having distribution functions G_{mc} and G_{lc}, respectively, we get

$$\mathbb{P}[Y_{\text{mc}} = 6\text{'}000] = \frac{2}{5}, \quad \mathbb{P}[Y_{\text{mc}} = 100\text{'}000] = \frac{2}{5} \quad \text{and} \quad \mathbb{P}[Y_{\text{mc}} = 500\text{'}000] = \frac{1}{5},$$

as well as

$$\mathbb{P}[Y_{\text{lc}} = 2\text{'}000\text{'}000] = \frac{1}{2}, \quad \mathbb{P}[Y_{\text{lc}} = 5\text{'}000\text{'}000] = \frac{1}{4} \quad \text{and} \quad \mathbb{P}[Y_{\text{lc}} = 10\text{'}000\text{'}000] = \frac{1}{4}.$$

(c) According to Theorem 2.14 of the lecture notes, S_{sc}, S_{mc} and S_{lc} are independent.

(d) In order to find $\mathbb{E}[S_{\text{sc}}]$, we need $\mathbb{E}[Y_{\text{sc}}]$, which can be calculated as

$$\mathbb{E}[Y_{\text{sc}}] = 100 \cdot \mathbb{P}[Y_{\text{sc}} = 100] + 300 \cdot \mathbb{P}[Y_{\text{mc}} = 300] + 500 \cdot \mathbb{P}[Y_{\text{lc}} = 500] = \frac{300}{10} + \frac{1200}{10} + \frac{1500}{10} = 300.$$

Now we can apply Proposition 2.11 of the lecture notes to get

$$\mathbb{E}[S_{\text{sc}}] = \frac{\lambda v}{2} \mathbb{E}[Y_{\text{sc}}] = 0.3 \cdot 300 = 90.$$

Similarly, we get

$$\mathbb{E}[Y_{\text{mc}}] = 142\text{'}400 \quad \text{and} \quad \mathbb{E}[Y_{\text{lc}}] = 4\text{'}750\text{'}000.$$

Thus, we find

$$\mathbb{E}[S_{\text{mc}}] = \frac{\lambda v}{3}\mathbb{E}[Y_{\text{mc}}] = 28\text{'}480 \quad \text{and} \quad \mathbb{E}[S_{\text{lc}}] = \frac{\lambda v}{6}\mathbb{E}[Y_{\text{lc}}] = 475\text{'}000.$$

Since $S = S_{\text{sc}} + S_{\text{mc}} + S_{\text{lc}}$, we get

$$\mathbb{E}[S] = \mathbb{E}[S_{\text{sc}}] + \mathbb{E}[S_{\text{mc}}] + \mathbb{E}[S_{\text{lc}}] = 503\text{'}570.$$

In order to find $\text{Var}(S_{\text{sc}})$, we need $\mathbb{E}[Y_{\text{sc}}^2]$, which can be calculated as

$$\begin{aligned}\mathbb{E}[Y_{\text{sc}}^2] &= 100^2 \cdot \mathbb{P}[Y_{\text{sc}} = 100] + 300^2 \cdot \mathbb{P}[Y_{\text{mc}} = 300] + 500^2 \cdot \mathbb{P}[Y_{\text{lc}} = 500] \\ &= \frac{30\text{'}000}{10} + \frac{360\text{'}000}{10} + \frac{750\text{'}000}{10} = 114\text{'}000.\end{aligned}$$

Now we can apply Proposition 2.11 of the lecture notes to get

$$\text{Var}(S_{\text{sc}}) = \frac{\lambda v}{2}\mathbb{E}[Y_{\text{sc}}^2] = 0.3 \cdot 114\text{'}000 = 34\text{'}200.$$

Similarly, we get

$$\mathbb{E}[Y_{\text{mc}}^2] = 54\text{'}014\text{'}400\text{'}000 \quad \text{and} \quad \mathbb{E}[Y_{\text{lc}}^2] = 33\text{'}250\text{'}000\text{'}000\text{'}000.$$

Thus, we find

$$\text{Var}(S_{\text{mc}}) = \frac{\lambda v}{3}\mathbb{E}[Y_{\text{mc}}^2] = 10\text{'}802\text{'}880\text{'}000 \quad \text{and} \quad \text{Var}(S_{\text{lc}}) = \frac{\lambda v}{6}\mathbb{E}[Y_{\text{lc}}^2] = 3\text{'}325\text{'}000\text{'}000\text{'}000.$$

Since $S = S_{\text{sc}} + S_{\text{mc}} + S_{\text{lc}}$ and S_{sc}, S_{mc} and S_{lc} are independent, we get

$$\sqrt{\text{Var}(S)} = \sqrt{\text{Var}(S_{\text{sc}}) + \text{Var}(S_{\text{mc}}) + \text{Var}(S_{\text{lc}})} = \sqrt{3\text{'}335\text{'}802\text{'}914\text{'}200} \approx 1\text{'}826\text{'}418.$$

(e) First, we define the random variable N_{lc} as

$$N_{\text{lc}} \sim \text{Poi}\left(\frac{\lambda v}{6}\right).$$

The probability that the total claim in the large claims layer exceeds 5 million can be calculated by looking at the complement, i.e. at the probability that the total claim in the large claims layer does not exceed 5 million. Since with three claims in the large claims layer we already exceed 5 million, it is enough to consider only up to two claims. Then, we get

$$\begin{aligned}\mathbb{P}\left[S_{\text{lc}} \leq 5\text{'}000\text{'}000\right] &= \mathbb{P}[N_{\text{lc}} = 0] + \mathbb{P}[N_{\text{lc}} = 1]\mathbb{P}[Y_{\text{lc}} \leq 5\text{'}000\text{'}000] + \mathbb{P}[N_{\text{lc}} = 2]\mathbb{P}[Y_{\text{lc}} = 2\text{'}000\text{'}000]^2 \\ &= \exp\left\{-\frac{\lambda v}{6}\right\} + \exp\left\{-\frac{\lambda v}{6}\right\}\frac{\lambda v}{6}\left(\frac{1}{2} + \frac{1}{4}\right) + \exp\left\{-\frac{\lambda v}{6}\right\}\left(\frac{\lambda v}{6}\right)^2 \frac{1}{2}\frac{1}{4} \\ &= \exp\left\{-0.1\right\}(1 + 0.075 + 0.00125) \\ &\approx 97.4\%.\end{aligned}$$

We can conclude that

$$\mathbb{P}\left[S_{\text{lc}} > 5\text{'}000\text{'}000\right] = 1 - \mathbb{P}\left[S_{\text{lc}} \leq 5\text{'}000\text{'}000\right] \approx 2.6\%.$$

Solution 4.4 Method of Moments

If $Y \sim \Gamma(\gamma, c)$, we have

$$\mathbb{E}[Y] = \frac{\gamma}{c} \quad \text{and} \quad \mathrm{Var}(Y) = \frac{\gamma}{c^2}.$$

We define the sample mean $\widehat{\mu}_8$ and the sample variance $\widehat{\sigma}_8^2$ of the eight observations $y_1, \ldots, y_8$ given on the exercise sheet as

$$\widehat{\mu}_8 = \frac{1}{8}\sum_{i=1}^{8} y_i = \frac{64}{8} = 8 \quad \text{and} \quad \widehat{\sigma}_8^2 = \frac{1}{7}\sum_{i=1}^{8}(y_i - \widehat{\mu}_8)^2 = \frac{28}{7} = 4.$$

The method of moments estimates $(\widehat{\gamma}, \widehat{c})$ of (γ, c) are defined to be those values that solve the equations

$$\widehat{\mu}_8 = \frac{\widehat{\gamma}}{\widehat{c}} \quad \text{and} \quad \widehat{\sigma}_8^2 = \frac{\widehat{\gamma}}{\widehat{c}^2}.$$

We see that $\widehat{\gamma} = \widehat{\mu}_8 \widehat{c}$ and, thus,

$$\widehat{\sigma}_8^2 = \frac{\widehat{\mu}_8 \widehat{c}}{\widehat{c}^2} = \frac{\widehat{\mu}_8}{\widehat{c}},$$

which is equivalent to

$$\widehat{c} = \frac{\widehat{\mu}_8}{\widehat{\sigma}_8^2} = \frac{8}{4} = 2.$$

Moreover, we get

$$\widehat{\gamma} = \frac{\widehat{\mu}_8^2}{\widehat{\sigma}_8^2} = \frac{64}{4} = 16.$$

We conclude that the method of moments estimates are given by $(\widehat{\gamma}, \widehat{c}) = (16, 2)$.