Non-Life Insurance: Mathematics and Statistics

Solution sheet 6

Solution 6.1 Goodness-of-Fit Test

Let Y be a random variable following a Pareto distribution with threshold $\theta = 200$ and tail index $\alpha = 1.25$. Then, the distribution function G of Y is given by

$$G(x) = 1 - \left(\frac{x}{\theta}\right)^{-\alpha} = 1 - \left(\frac{x}{200}\right)^{-1.25},$$

for all $x \geq \theta$. For example for the interval I_2 we then have

$$\mathbb{P}[Y \in I_2] = \mathbb{P}[239 \leq Y < 301] = G(301) - G(239) = 1 - \left(\frac{301}{200}\right)^{-1.25} - \left[1 - \left(\frac{239}{200}\right)^{-1.25}\right] \approx 0.2.$$

By analogous calculations for the other four intervals, we get

$$\mathbb{P}[Y \in I_1] = \mathbb{P}[Y \in I_2] = \mathbb{P}[Y \in I_3] = \mathbb{P}[Y \in I_4] = \mathbb{P}[Y \in I_5] \approx 0.2.$$

Let E_i and O_i denote the expected number of observations in I_i and the observed number of observations in I_i, respectively, for all $i \in \{1, \ldots, 5\}$. As we have 20 observations in our data, we can calculate for example E_2 as

$$E_2 = 20 \cdot \mathbb{P}[Y \in I_2] \approx 4.$$

The values of the expected number of observations and the observed number of observations in the five intervals as well as their squared differences are summarized in the following table:

i	1	2	3	4	5
O_i	4	0	8	6	2
E_i	4	4	4	4	4
$(O_i - E_i)^2$	0	16	16	4	4

Table 1: Observed number of observations and expected number of observations in the five intervals as well as their squared differences.

The test statistic of the χ^2-goodness-of-fit test using 5 intervals and 20 observations is given by

$$X^2_{20,5} = \sum_{i=1}^{5} \frac{(O_i - E_i)^2}{E_i} = \frac{0}{4} + \frac{16}{4} + \frac{16}{4} + \frac{4}{4} + \frac{4}{4} = 10.$$

Let $\alpha = 5\%$. Then, the $(1-\alpha)$-quantile of the χ^2-distribution with $5 - 1 = 4$ degrees of freedom is given by approximately 9.49. Since this is smaller than $X^2_{20,5}$, we can reject the null hypothesis of having a Pareto distribution with threshold $\theta = 200$ and tail index $\alpha = 1.25$ as claim size distribution at the significance level of 5%.

Solution 6.2 Log-Normal Distribution and Deductible

(a) Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then, the moment generating function M_X of X is given by

$$M_X(r) = \mathbb{E}\left[\exp\{rX\}\right] = \exp\left\{r\mu + \frac{r^2\sigma^2}{2}\right\},$$

for all $r \in \mathbb{R}$. Since Y_1 has a log-normal distribution with mean parameter μ and variance parameter σ^2, we have

$$Y_1 \overset{\mathrm{d}}{=} \exp\{X\}.$$

Hence, the expectation, the variance and the coefficient of variation of Y_1 can be calculated as

$$\begin{aligned}
\mathbb{E}[Y_1] &= \mathbb{E}\left[\exp\{X\}\right] = \mathbb{E}\left[\exp\{1 \cdot X\}\right] = M_X(1) = \exp\left\{\mu + \frac{\sigma^2}{2}\right\},\\
\operatorname{Var}(Y_1) &= \mathbb{E}[Y_1^2] - \mathbb{E}[Y_1]^2 = \mathbb{E}\left[\exp\{2X\}\right] - M_X(1)^2 = M_X(2) - M_X(1)^2\\
&= \exp\left\{2\mu + \frac{4\sigma^2}{2}\right\} - \exp\left\{2\mu + 2\frac{\sigma^2}{2}\right\} = \exp\left\{2\mu + \sigma^2\right\}\left(\exp\left\{\sigma^2\right\} - 1\right) \quad \text{and}\\
\operatorname{Vco}(Y_1) &= \frac{\sqrt{\operatorname{Var}(Y_1)}}{\mathbb{E}[Y_1]} = \frac{\exp\left\{\mu + \sigma^2/2\right\}\sqrt{\exp\left\{\sigma^2\right\} - 1}}{\exp\left\{\mu + \sigma^2/2\right\}} = \sqrt{\exp\left\{\sigma^2\right\} - 1}.
\end{aligned}$$

(b) From part (a) we know that

$$\begin{aligned}
\sigma &= \sqrt{\log[\operatorname{Vco}(Y_1)^2 + 1]} \quad \text{and}\\
\mu &= \log \mathbb{E}[Y_1] - \frac{\sigma^2}{2}.
\end{aligned}$$

Since $\mathbb{E}[Y_1] = 3\text{'}000$ and $\operatorname{Vco}(Y_1) = 4$, we get

$$\begin{aligned}
\sigma &= \sqrt{\log(4^2 + 1)} \approx 1.68 \quad \text{and}\\
\mu &\approx \log 3\text{'}000 - \frac{(1.68)^2}{2} \approx 6.59.
\end{aligned}$$

(i) The claims frequency λ is given by $\lambda = \mathbb{E}[N]/v$. With the introduction of the deductible $d = 500$, the number of claims changes to

$$N^{\text{new}} = \sum_{i=1}^{N} 1_{\{Y_i > d\}}.$$

Using the independence of N and $Y_1, Y_2, \ldots$, we get

$$\mathbb{E}[N^{\text{new}}] = \mathbb{E}\left[\sum_{i=1}^{N} 1_{\{Y_i > d\}}\right] = \mathbb{E}[N]\mathbb{E}[1_{\{Y_1 > d\}}] = \mathbb{E}[N]\mathbb{P}[Y_1 > d].$$

Let Φ denote the distribution function of a standard Gaussian distribution. Since $\log Y_1$ has a Gaussian distribution with mean μ and variance σ^2, we have

$$\mathbb{P}[Y_1 > d] = 1 - \mathbb{P}\left[\frac{\log Y_1 - \mu}{\sigma} \leq \frac{\log d - \mu}{\sigma}\right] = 1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right).$$

Hence, the new claims frequency λ^{new} is given by

$$\lambda^{\text{new}} = \mathbb{E}[N^{\text{new}}]/v = \mathbb{E}[N]\mathbb{P}[Y_1 > d]/v = \lambda\mathbb{P}[Y_1 > d] = \lambda\left[1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right)\right].$$

Inserting the values of d, μ and σ, we get

$$\lambda^{\text{new}} \approx \lambda \left[1 - \Phi\left(\frac{\log 500 - 6.59}{1.68}\right)\right] \approx 0.59 \cdot \lambda.$$

Note that the introduction of this deductible reduces the administrative burden a lot, because we expect that 41% of the claims disappear.

(ii) With the introduction of the deductible $d = 500$, the claim sizes change to

$$Y_i^{\text{new}} = Y_i - d \,|\, Y_i > d.$$

Thus, the new expected claim size is given by

$$\mathbb{E}[Y_1^{\text{new}}] = \mathbb{E}[Y_1 - d | Y_1 > d] = e(d),$$

where $e(d)$ is the mean excess function of Y_1 above d. According to the lecture notes, $e(d)$ is given by

$$e(d) = \mathbb{E}[Y_1] \left[\frac{1 - \Phi\left(\frac{\log d - \mu - \sigma^2}{\sigma}\right)}{1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right)}\right] - d.$$

Inserting the values of d, μ, σ and $\mathbb{E}[Y_1]$, we get

$$\mathbb{E}[Y_1^{\text{new}}] \approx 3\text{'}000 \left[\frac{1 - \Phi\left(\frac{\log 500 - 6.59 - 1.68^2}{1.68}\right)}{1 - \Phi\left(\frac{\log 500 - 6.59}{1.68}\right)}\right] - 500 \approx 4\text{'}456 \approx 1.49 \cdot \mathbb{E}[Y_1].$$

(iii) According to Proposition 2.2 of the lecture notes, the expected total claim amount $\mathbb{E}[S]$ is given by

$$\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[Y_1].$$

With the introduction of the deductible $d = 500$, the total claim amount S changes to S^{new}, which can be written as

$$S^{\text{new}} = \sum_{i=1}^{N^{\text{new}}} Y_i^{\text{new}}.$$

Hence, the expected total claim amount changes to

$$\begin{aligned}
\mathbb{E}\left[S^{\text{new}}\right] &= \mathbb{E}\left[N^{\text{new}}\right] \mathbb{E}\left[Y_1^{\text{new}}\right] \\
&= \mathbb{E}[N]\mathbb{P}[Y_1 > d] e(d) \\
&= \lambda v \left[1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right)\right] \cdot \left(\mathbb{E}[Y_1] \left[\frac{1 - \Phi\left(\frac{\log d - \mu - \sigma^2}{\sigma}\right)}{1 - \Phi\left(\frac{\log d - \mu}{\sigma}\right)}\right] - d\right).
\end{aligned}$$

Inserting the values of d, μ, σ and $\mathbb{E}[Y_1]$, we get

$$\begin{aligned}
\mathbb{E}\left[S^{\text{new}}\right] &\approx \lambda v \left[1 - \Phi\left(\frac{\log 500 - 6.59}{1.68}\right)\right] \cdot \left(3\text{'}000 \left[\frac{1 - \Phi\left(\frac{\log 500 - 6.59 - 1.68^2}{1.68}\right)}{1 - \Phi\left(\frac{\log 500 - 6.59}{1.68}\right)}\right] - 500\right) \\
&\approx \lambda v \cdot 0.59 \cdot 4\text{'}456 \\
&= 0.88 \cdot \mathbb{E}[S].
\end{aligned}$$

In particular, the insurance company can grant a discount of roughly 12% on the pure risk premium. Note that also the administrative expenses on claims handling will reduce substantially because we only have 59% of the original claims, see the result in (i).

Solution 6.3 Re-Insurance Covers and Leverage Effect

(a) From formula (3.10) of the lecture notes, we have

$$\mathbb{E}[(Y-d)_+] = \mathbb{P}[Y > d](\mathbb{E}[Y|Y > d] - d).$$

Note that a gamma distribution with shape parameter equal to 1 is an exponential distribution. The characteristic property of an exponential distribution is the so-called memorylessness property

$$\mathbb{P}[Y > t+s|Y > t] = \mathbb{P}[Y > s],$$

for all $t, s > 0$. In particular, this property leads to (see also below)

$$\mathbb{E}[Y|Y > d] = \mathbb{E}[Y] + d,$$

which, in turn, implies

$$\mathbb{E}[(Y-d)_+] = \mathbb{P}[Y > d]\mathbb{E}[Y].$$

Indeed, we have

$$\begin{aligned}
\mathbb{E}[Y|Y > d] &= \frac{\mathbb{E}[Y 1_{\{Y>d\}}]}{\mathbb{P}[Y > d]} = \frac{1}{\mathbb{P}[Y > d]} \int_0^\infty y 1_{\{y>d\}} \frac{1}{400} \exp\left\{-\frac{y}{400}\right\} dy \\
&= \frac{1}{\exp\left\{-\frac{d}{400}\right\}} \int_d^\infty y \frac{1}{400} \exp\left\{-\frac{y}{400}\right\} dy \\
&= \exp\left\{\frac{d}{400}\right\} \int_0^\infty (u+d) \frac{1}{400} \exp\left\{-\frac{u}{400}\right\} \exp\left\{-\frac{d}{400}\right\} du \\
&= \int_0^\infty u \frac{1}{400} \exp\left\{-\frac{u}{400}\right\} du + d \int_0^\infty \frac{1}{400} \exp\left\{-\frac{u}{400}\right\} du \\
&= \mathbb{E}[Y] + d,
\end{aligned}$$

where in the third equality we used the substitution $u = y - d$.

(b) By looking at the graphs in Figure 1 on the exercise sheet, we find the following re-insurance covers:

(i) $(Y - 200)_+$,
(ii) $\min\{Y, 400\}$,
(iii) $\max\{\min\{Y, 200\}, Y - 200\}$.

(c) (i) Using part (a), we get

$$\mathbb{E}[(Y-200)_+] = \mathbb{P}[Y > 200]\mathbb{E}[Y] = \exp\left\{-\frac{200}{400}\right\} 400 = \frac{400}{\sqrt{\exp\{1\}}} \approx 243.$$

(ii) According to the lecture notes, we have the identity

$$\mathbb{E}[\min\{Y, 400\}] = \mathbb{E}[Y] - \mathbb{E}[(Y-400)_+].$$

Again using part (a), we get

$$\begin{aligned}
\mathbb{E}[\min\{Y, 400\}] &= \mathbb{E}[Y] - \mathbb{P}[Y > 400]\mathbb{E}[Y] = 400\left(1 - \exp\left\{-\frac{400}{400}\right\}\right) \\
&= 400(1 - \exp\{-1\}) \approx 253.
\end{aligned}$$

(iii) We can write

$$\max\{\min\{Y,200\}, Y-200\} = \begin{cases} Y, & \text{if } 0 \le Y < 200, \\ 200, & \text{if } 200 \le Y < 400, \\ Y-200, & \text{if } Y \ge 400. \end{cases}$$

This leads to

$$\begin{aligned} \max\{\min\{Y,200\}, Y-200\} &= Y + (200-Y)\cdot 1_{\{200\le Y<400\}} - 200\cdot 1_{\{Y\ge 400\}} \\ &= Y + (200-Y)\cdot 1_{\{Y\ge 200\}} - (200-Y)\cdot 1_{\{Y\ge 400\}} \\ &\quad - 200\cdot 1_{\{Y\ge 400\}} \\ &= Y - (Y-200)\cdot 1_{\{Y\ge 200\}} + (Y-400)\cdot 1_{\{Y\ge 400\}} \\ &= Y - (Y-200)_+ + (Y-400)_+. \end{aligned}$$

Using part (a), the expected value is then given by

$$\begin{aligned} \mathbb{E}[\max\{\min\{Y,200\}, Y-200\}] &= \mathbb{E}[Y] - \mathbb{E}[(Y-200)_+] + \mathbb{E}[(Y-400)_+] \\ &= \mathbb{E}[Y] - \mathbb{P}[Y>200]\mathbb{E}[Y] + \mathbb{P}[Y>400]\mathbb{E}[Y] \\ &= 400\left(1 - \exp\left\{-\frac{1}{2}\right\} + \exp\{-1\}\right) \\ &\approx 305. \end{aligned}$$

(d) We have

$$Y_0 \sim \Gamma\left(1, \frac{1}{400}\right).$$

The formula (3.5) of the lecture notes then implies

$$Y_1 \overset{(d)}{=} (1+i)Y_0 \sim \Gamma\left(1, \frac{1}{400(1+i)}\right).$$

Using part (a), we get

$$\mathbb{E}[(Y_1-d)_+] = \mathbb{P}[Y_1>d]\mathbb{E}[Y_1] = \exp\left\{-\frac{d}{400(1+i)}\right\}400(1+i)$$

and

$$(1+i)\mathbb{E}[(Y_0-d)_+] = (1+i)\mathbb{P}[Y_0>d]\mathbb{E}[Y_0] = (1+i)\exp\left\{-\frac{d}{400}\right\}400.$$

We get

$$\frac{\mathbb{E}[(Y_1-d)_+]}{(1+i)\mathbb{E}[(Y_0-d)_+]} = \frac{\exp\left\{-\frac{d}{400(1+i)}\right\}}{\exp\left\{-\frac{d}{400}\right\}} = \exp\left\{\frac{d}{400}\left(1-\frac{1}{1+i}\right)\right\} > 1,$$

since $i>0$. We conclude that

$$\mathbb{E}[(Y_1-d)_+] > (1+i)\mathbb{E}[(Y_0-d)_+].$$

The reason for this (strict) inequality, which is called leverage effect, is that not only the claim sizes are growing under inflation, but also the number of claims that exceed the threshold d increases under inflation, as we do not adapt the threshold d to inflation.

Solution 6.4 Inflation and Deductible

Let Y be a random variable following a Pareto distribution with threshold $\theta > 0$ and tail index $\alpha > 1$. Then, the expectation $\mathbb{E}[Y]$ of Y and the mean excess function $e_Y(u)$ of Y above $u > \theta$ are given by

$$\mathbb{E}[Y] = \frac{\alpha}{\alpha-1}\theta \quad \text{and} \quad e_Y(u) = \frac{1}{\alpha-1}u.$$

Since the insurance company only has to pay the part that exceeds the deductible θ, this year's average claim payment z is

$$z = \mathbb{E}[Y] - \theta = \frac{\alpha}{\alpha-1}\theta - \theta = \frac{\theta}{\alpha-1}.$$

For the total claim size $\widetilde{Y}$ of a claim next year we have

$$\widetilde{Y} \stackrel{d}{=} (1+r)Y \sim \text{Pareto}([1+r]\theta, \alpha).$$

Let $\rho\theta$ for some $\rho > 0$ denote the increase of the deductible that is needed such that the average claim payment remains unchanged. Then, next year's average claim payment is given by

$$\widetilde{z} = \mathbb{E}\left[\left(\widetilde{Y} - [1+\rho]\theta\right)_+\right].$$

Let's first assume that we can choose a $\rho < r$ such that $z = \widetilde{z}$. In this case we get

$$\widetilde{Y} \geq (1+r)\theta \quad \text{a.s.} \qquad \Longrightarrow \qquad \widetilde{Y} \geq (1+\rho)\theta \quad \text{a.s.}$$

and, thus,

$$\widetilde{z} = \mathbb{E}\left[\widetilde{Y} - (1+\rho)\theta\right] = \mathbb{E}\left[\widetilde{Y}\right] - (1+\rho)\theta = \frac{\alpha}{\alpha-1}(1+r)\theta - (1+\rho)\theta.$$

We have $z = \widetilde{z}$ if and only if

$$\begin{aligned} \frac{\alpha}{\alpha-1}\theta - \theta = \frac{\alpha}{\alpha-1}(1+r)\theta - (1+\rho)\theta \qquad &\Longleftrightarrow \qquad 0 = \frac{\alpha}{\alpha-1}r\theta - \rho\theta \\ &\Longleftrightarrow \qquad \rho = \frac{\alpha}{\alpha-1}r > r, \end{aligned}$$

which is a contradiction to the assumption $\rho < r$. Hence, we conclude that $\rho \geq r$, i.e. the percentage increase in the deductible has to be bigger than the inflation. Assuming $\rho \geq r$, we can calculate

$$\begin{aligned} \widetilde{z} &= \mathbb{E}\left[\left(\widetilde{Y} - [1+\rho]\theta\right) \cdot 1_{\left\{\widetilde{Y} - (1+\rho)\theta > 0\right\}}\right] \\ &= \mathbb{E}\left[\widetilde{Y} - (1+\rho)\theta \,\middle|\, \widetilde{Y} > (1+\rho)\theta\right] \cdot \mathbb{P}\left[\widetilde{Y} > (1+\rho)\theta\right] \\ &= e_{\widetilde{Y}}([1+\rho]\theta) \cdot \mathbb{P}\left[\widetilde{Y} > (1+\rho)\theta\right] \\ &= \frac{1}{\alpha-1}(1+\rho)\theta \cdot \left[\frac{(1+\rho)\theta}{(1+r)\theta}\right]^{-\alpha} \\ &= \frac{\theta}{\alpha-1}(1+r)^{\alpha}(1+\rho)^{-\alpha+1} \\ &= z \cdot (1+r)^{\alpha}(1+\rho)^{-\alpha+1}. \end{aligned}$$

We have $z = \widetilde{z}$ if and only if

$$(1+r)^{\alpha}(1+\rho)^{-\alpha+1} = 1 \quad \Longleftrightarrow \quad \rho = (1+r)^{\frac{\alpha}{\alpha-1}} - 1.$$

We conclude that if we want the average claim payment to remain unchanged, we have to increase the deductible θ by the amount

$$\theta\left[(1+r)^{\frac{\alpha}{\alpha-1}} - 1\right].$$