Appendix B: A Komlós-type lemma from probability theory

These notes provide a formulation and proof for an elementary lemma from probability theory which is extremely useful in many optimisation problems involving convexity. Recall that \( L^0 \) denotes the vector space of all (equivalence classes of, for the equivalence relation of equality \( P \)-a.s.) random variables on a given probability space \( (\Omega, \mathcal{F}, P) \) and taking values in \( \mathbb{R} \).

For a sequence \((Y_n)_{n \in \mathbb{N}}\) in \( L^0 \), we denote for \( m \in \mathbb{N} \) by \( \text{conv}(Y_m, Y_{m+1}, \ldots) \) the set of all (finite) convex combinations of \((Y_k)_{k \geq m}\), i.e. all \( Y \) of the form \( Y = \sum_{k=m}^{\infty} \lambda_k Y_k \) with the \( \lambda_k \geq 0 \) satisfying \( \sum_{k=m}^{\infty} \lambda_k = 1 \) and at most finally many \( \lambda_k \neq 0 \).

**Lemma B.1.** For any sequence \((Y_n)_{n \in \mathbb{N}}\) of nonnegative random variables, there exists a sequence \((\tilde{Y}_n)_{n \in \mathbb{N}}\) with \( \tilde{Y}_n \in \text{conv}(Y_n, Y_{n+1}, \ldots) \) for all \( n \) and \( \tilde{Y}_n \to \tilde{Y}_\infty \) \( P \)-a.s. for some random variable \( \tilde{Y}_\infty \) taking values in \([0, +\infty)\). Moreover, if \( P[Y_n \geq \alpha] \geq \delta > 0 \) for some \( \alpha > 0 \), then \( P[\tilde{Y}_\infty > 0] > 0 \). If \( \text{conv}(Y_1, Y_2, \ldots) \) is bounded in \( L^0 \), then \( \tilde{Y}_\infty < \infty \) \( P \)-a.s.

**Proof.** Set \( C_n := \text{conv}(Y_n, Y_{n+1}, \ldots) \supseteq C_{n+1} \) so that the sequence \( J_n := \inf_{Y \in C_n} E[e^{-Y}] \) increases to some \( J \leq 1 \). Take a sequence \((Y'_n)_{n \in \mathbb{N}}\) with \( Y'_n \in C_n \) and \( E[e^{-Y'_n}] \leq J_n + \frac{1}{n} \) for all \( n \). For \( \varepsilon > 0 \), define the set

\[
B_\varepsilon := \left\{(x, y) \in [0, \infty)^2 : |x - y| \geq \varepsilon \text{ and } x \wedge y \leq \frac{1}{\varepsilon}\right\}
\]

(a picture will help to illustrate this). As the mapping \( z \mapsto e^{-z} \) is convex, we always have

\[
e^{-\frac{(x+y)}{2}} \leq \frac{1}{2}(e^{-x} + e^{-y}).
\]

For \( (x, y) \in B_\varepsilon \), a calculation gives

\[
e^{-\frac{(x+y)}{2}} - \frac{1}{2}(e^{-x} + e^{-y}) \leq -\delta \quad \text{for some } \delta = \delta(\varepsilon) > 0,
\]

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and therefore
\[ e^{-(x+y)/2} \leq \frac{1}{2}(e^{-x} + e^{-y}) - \delta I_{B_\epsilon}(x,y). \]

Choosing \( x := Y'_m \) and \( y := Y'_n \) yields for \( n \neq m \) that
\[ J_{n \wedge m} \leq E \left[ e^{-(Y'_m + Y'_n)/2} \right] \]
\[ \leq \frac{1}{2} \left( E\left[ e^{-Y'_m} \right] + E\left[ e^{-Y'_n} \right] \right) - \delta P[(Y'_m, Y'_n) \in B_\epsilon] \]
\[ \leq \frac{1}{2} \left( J_m + \frac{1}{m} + J_n + \frac{1}{n} \right) - \delta P[(Y'_m, Y'_n) \in B_\epsilon], \]
and so we obtain that
\[ \lim_{n,m \to \infty} P[(Y'_m, Y'_n) \in B_\epsilon] = 0. \]

Considering the separate cases \(|x - y| < \epsilon\) or \(x \wedge y > \frac{1}{\epsilon}\) or \((x, y) \in B_\epsilon\) leads to the estimate
\[ |e^{-x} - e^{-y}| \leq \epsilon + 2e^{-1/\epsilon} + 2I_{B_\epsilon}(x,y). \]

This gives in turn that
\[ \left| E\left[ e^{-Y'_m} - e^{-Y'_n} \right] \right| \leq \epsilon + 2e^{-1/\epsilon} + 2P[(Y'_m, Y'_n) \in B_\epsilon] \]
so that \((e^{-Y'_n})_{n \in \mathbb{N}}\) is a Cauchy sequence in \(L^1(P)\) and hence convergent in \(L^1(P)\). Therefore this sequence has a subsequence \((e^{-\tilde{Y}_n})_{n \in \mathbb{N}}\) which converges \(P\)-a.s., and then the sequence \((\tilde{Y}_n)_{n \in \mathbb{N}}\) is also \(P\)-a.s. convergent and has \(\tilde{Y}_n \in C_n\) like for \(Y'_n\).

If \(P[Y_n \geq \alpha] \geq \delta > 0\), then \(E[e^{-Y_n}] \leq 1 - \delta + \delta e^{-\delta}\) and the same bound < 1 holds for any \(\tilde{Y}_n \in C_n\) by Jensen's inequality. So \(E[e^{-\tilde{Y}_\infty}] \leq 1 - \delta + \delta e^{-\delta}\) by dominated convergence, and so \(P[\tilde{Y}_\infty > 0] > 0\).

Finally, if a set is bounded in \(L^0\), all its accumulation points in \(L^0\) are finite-valued \(P\)-a.s. \(\rightarrow\) exercise. \q.e.d.

**Remark.** If one has extra properties for the original sequence \((Y_n)_{n \in \mathbb{N}}\), one can also say more about the limit \(\tilde{Y}_\infty\). For example, if all the \(Y_n\) are bounded by some constant, the same is true for the \(\tilde{Y}_n\) and hence also for \(\tilde{Y}_\infty\).