Appendix C: Essential supremum and infimum

These notes briefly recall the definition and main properties of the essential supremum and infimum of a family of (possibly extended) real-valued random variables. We fix a probability space $(\Omega, \mathcal{F}, P)$, an arbitrary index set $\Lambda \neq \emptyset$ and a family $(Y_\lambda)_{\lambda \in \Lambda}$ of (possibly extended) real-valued random variables on $(\Omega, \mathcal{F}, P)$.

Definition. A random variable Z is called *essential supremum* of the family $(Y_\lambda)_{\lambda \in \Lambda}$ if

(i) $Z \geq Y_\lambda$ P-a.s. for each $\lambda \in \Lambda$.

(ii) $Z \leq Z'$ P-a.s. for each random variable Z' satisfying $Z' \geq Y_\lambda$ P-a.s. for each $\lambda \in \Lambda$.

We then write briefly $Z = \operatorname*{ess\,sup}_{\lambda \in \Lambda} Y_\lambda$. The *essential infimum* $\operatorname*{ess\,inf}_{\lambda \in \Lambda} Y_\lambda$ is defined analogously by simply reversing all inequalities above.

Remarks. 1) If Λ is countable, we can take the pointwise supremum $Z(\omega) := \sup_{\lambda \in \Lambda} Y_\lambda(\omega)$; this is measurable and thus a random variable. But if Λ is uncountable, this no longer works; on the one hand, the pointwise supremum may fail to be measurable, and on the other hand, (i) and (ii) can also fail, as illustrated by the subsequent example.

2) By (ii), an essential supremum is P-a.s. unique; so we only have to prove its existence.

3) The subsequent results can of course also be formulated and proved (with obvious changes) for the essential infimum instead of supremum.

4) Since the definition and all the arguments below only involve the order structure of $\mathbb{R}$, but not the actual values of the random variables under consideration, everything works equally well if we allow the Y_λ to take values in $[-\infty, +\infty]$.

Example. Let $\Omega = [0,1]$, $P =$ Lebesgue measure, $\Lambda = [0,1]$ and $Y_\lambda(\omega) = I_{\{\lambda\}}(\omega)$. Then

$$\sup_{\lambda \in \Lambda} Y_\lambda(\omega) = 1 \qquad \text{for each fixed } \omega,$$

and so the pointwise supremum $\sup_{\lambda\in\Lambda} Y_\lambda \equiv 1$ is here measurable. But for every fixed λ, we also have $Y_\lambda = 0$ P-a.s. and thus obviously

$$\operatorname*{ess\,sup}_{\lambda\in\Lambda} Y_\lambda = 0 \qquad (P\text{-a.s.})$$

Proposition C.1. *For any family $(Y_\lambda)_{\lambda\in\Lambda}$ of (possibly extended) real-valued random variables, $\operatorname*{ess\,sup}_{\lambda\in\Lambda} Y_\lambda = Z$ exists, and $Z = \sup_{j\in J_0} Y_j$ for some countable subset J_0 of Λ.*

Proof. Since the above definition only involves the order structure of $I\!R$, we may and do assume without loss of generality that all Y_λ are bounded, uniformly in λ and ω. Set

$$c := \sup\left\{ E\left[\sup_{j\in J} Y_j\right] : J \subseteq \Lambda \text{ countable}\right\}$$

and choose a sequence $(J_n)_{n\in I\!N}$ of countable subsets of Λ such that

$$\lim_{n\to\infty} E\left[\sup_{j\in J_n} Y_j\right] = c.$$

Then $J_0 := \bigcup_{n\in I\!N} J_n \subseteq \Lambda$ is countable, so $Z := \sup_{j\in J_0} Y_j$ is a random variable, and $E[Z] = c$ by monotone integration. We claim that Z does the job, and so we check the required properties.

(ii) If $Z' \geq Y_\lambda$ P-a.s. for each $\lambda \in \Lambda$, then also $P\left[Z' \geq Y_j \text{ for all } j \in J_0\right] = 1$ because J_0 is countable, and thus $Z' \geq Z$ P-a.s. by the definition of Z.

(i) For each $\lambda \in \Lambda$, we have $Z \vee Y_\lambda = \max(Z, Y_\lambda) \geq Z$, and by the definitions of c and J_0,

$$E\left[Z \vee Y_\lambda\right] = E\left[\sup_{j\in J_0\cup\{\lambda\}} Y_j\right] \leq c = E[Z].$$

Hence $Z \vee Y_\lambda - Z \geq 0$ and $E\left[Z \vee Y_\lambda - Z\right] \leq 0$; so we must have $Z \vee Y_\lambda = Z$ P-a.s., and thus $Z \geq Y_\lambda$ P-a.s. This holds for each $\lambda \in \Lambda$, and so Z satisfies (i). **q.e.d.**

Corollary C.2. *Suppose that $(Y_\lambda)_{\lambda\in\Lambda}$ is directed upward, i.e., for each pair λ, λ' in Λ, there is some $\mu \in \Lambda$ such that $\max(Y_\lambda, Y_{\lambda'}) \le Y_\mu$; this holds in particular if the family $(Y_\lambda)_{\lambda\in\Lambda}$ is closed under taking maxima. Then there is a sequence $(j_n)_{n\in\mathbb{N}}$ in Λ such that*

$$\operatorname*{ess\,sup}_{\lambda\in\Lambda} Y_\lambda = \nearrow - \lim_{n\to\infty} Y_{j_n} \qquad P\text{-a.s.},$$

i.e., $Y_{j_n} \le Y_{j_{n+1}}$ P-a.s. for each n and $Y_{j_n} \nearrow \operatorname*{ess\,sup}_{\lambda\in\Lambda} Y_\lambda$ P-a.s.

Proof. Choose $J_0 = \{\lambda_n : n \in \mathbb{N}\} \subseteq \Lambda$ countable with $\operatorname*{ess\,sup}_{\lambda\in\Lambda} Y_\lambda = \sup_{n\in\mathbb{N}} Y_{\lambda_n}$. Set $j_1 := \lambda_1$ and choose recursively an element j_n of Λ such that $\max\left(Y_{j_{n-1}}, Y_{\lambda_n}\right) \le Y_{j_n}$. Then clearly

$$Y_{j_{n-1}} \le Y_{j_n} \le \operatorname*{ess\,sup}_{\lambda\in\Lambda} Y_\lambda \qquad P\text{-a.s. for all } n,$$

and induction yields

$$Y_{j_n} \ge \max_{k=1,\ldots,n} Y_{\lambda_k},$$

so that

$$\operatorname*{ess\,sup}_{\lambda\in\Lambda} Y_\lambda \ge \nearrow - \lim_{n\to\infty} Y_{j_n} = \sup_{n\in\mathbb{N}} Y_{j_n} \ge \sup_{n\in\mathbb{N}} Y_{\lambda_n} = \operatorname*{ess\,sup}_{\lambda\in\Lambda} Y_\lambda.$$

This gives the assertion. **q.e.d.**