Appendix D: The bipolar theorem

These notes provide a formulation of the bipolar theorem from functional analysis. We formulate the result here for the setting we need, which means that we use the dual pair $(L^\infty, L^1)$ with the duality pairing given by $(Z, Y) = E[ZY]$ for $Z \in L^\infty$ and $Y \in L^1$.

For a subset $C \subseteq L^\infty$, the polar of $C$ in $L^1$ is

$$C^\circ := \{Y \in L^1 : (Z, Y) \leq 1 \text{ for all } Z \in C\}.$$ 

In the same way, the polar in $L^\infty$ of $D \subseteq L^1$ is

$$D^\circ := \{Z \in L^\infty : (Z, Y) \leq 1 \text{ for all } Y \in D\}.$$ 

The bipolar of $C \subseteq L^\infty$ is then the polar of $C^\circ$,

$$C^{\circ \circ} := (C^\circ)^\circ \subseteq L^\infty.$$ 

It is easy to check that for any $D \subseteq L^1$, the polar $D^\circ$ is a convex set in $L^\infty$, that $0 \in D^\circ$ and that $D^\circ$ is $\sigma(L^\infty, L^1)$-closed, i.e. weak* closed in $L^\infty$. If $C \subseteq L^\infty$ is a cone with vertex at 0 (meaning that $\lambda C \subseteq C$ for all $\lambda > 0$), then we also have

$$C^\circ = \{Y \in L^1 : (Z, Y) \leq 0 \text{ for all } Z \in C\};$$ 

so $C^\circ$ is then also a cone with vertex at 0, and hence

$$C^{\circ \circ} = \{Z \in L^\infty : (Z, Y) \leq 0 \text{ for all } Y \in C^\circ\}.$$ 

**Theorem D.1. (Bipolar theorem)** For any $C \subseteq L^\infty$, its bipolar $C^{\circ \circ}$ is the $\sigma(L^\infty, L^1)$-closed convex hull of $C \cup \{0\}$, i.e., the smallest convex and weak* closed subset of $L^\infty$ containing $C$ and 0.

In particular, if $C$ is a convex cone with vertex at 0, then $C^\circ$ is the weak* closure of $C$; if in addition $C$ is weak* closed, then $C^{\circ \circ} = C$.

**Proof.** See [1, Theorem IV.1.5].
Remark. While the above result looks simple, it is not quite straightforward. In fact, the argument for showing that the bipolar $C^{\circ\circ}$ is contained in the $\sigma(L^\infty, L^1)$-closed convex hull of $C \cup \{0\}$ uses the separation theorem for convex sets and is thus based on the Hahn–Banach theorem.

Reference