Appendix D: The bipolar theorem

These notes provide a formulation of the bipolar theorem from functional analysis. We formulate the result here for the setting we need, which means that we use the dual pair (L^∞, L^1) with the duality pairing given by $(Z, Y) = E[ZY]$ for $Z \in L^\infty$ and $Y \in L^1$.

For a subset $C \subseteq L^\infty$, the *polar* of C in L^1 is

$$C^\circ := \{Y \in L^1 : (Z, Y) \leq 1 \text{ for all } Z \in C\}.$$

In the same way, the polar in L^∞ of $D \subseteq L^1$ is

$$D^\circ := \{Z \in L^\infty : (Z, Y) \leq 1 \text{ for all } Y \in D\}.$$

The *bipolar* of $C \subseteq L^\infty$ is then the polar of C°,

$$C^{\circ\circ} := (C^\circ)^\circ \subseteq L^\infty.$$

It is easy to check that for any $D \subseteq L^1$, the polar D° is a convex set in L^∞, that $0 \in D^\circ$ and that D° is $\sigma(L^\infty, L^1)$-closed, i.e. weak* closed in L^∞. If $C \subseteq L^\infty$ is a cone with vertex at 0 (meaning that $\lambda C \subseteq C$ for all $\lambda > 0$), then we also have

$$C^\circ = \{Y \in L^1 : (Z, Y) \leq 0 \text{ for all } Z \in C\};$$

so C° is then also a cone with vertex at 0, and hence

$$C^{\circ\circ} = \{Z \in L^\infty : (Z, Y) \leq 0 \text{ for all } Y \in C^\circ\}.$$

Theorem D.1. (Bipolar theorem) *For any $C \subseteq L^\infty$, its bipolar $C^{\circ\circ}$ is the $\sigma(L^\infty, L^1)$-closed convex hull of $C \cup \{0\}$, i.e., the smallest convex and weak* closed subset of L^∞ containing C and 0.*

In particular, if C is a convex cone with vertex at 0, then $C^{\circ\circ}$ is the weak closure of C; if in addition C is weak* closed, then $C^{\circ\circ} = C$.*

Proof. See [1, Theorem IV.1.5].

Remark. While the above result looks simple, it is not quite straightforward. In fact, the argument for showing that the bipolar $C^{\circ\circ}$ is contained in the $\sigma(L^\infty, L^1)$-closed convex hull of $C \cup \{0\}$ uses the separation theorem for convex sets and is thus based on the Hahn–Banach theorem.

Reference

[1] H. H. Schaefer (with M. P. Wolff) (1999), "Topological Vector Spaces", *second edition, Springer*