Mathematical Finance

Exercise sheet 1

Please hand in your solutions by Wednesday, 26.09.2018, 12:00 into your assistant's box next to HG G 53.2.

Exercise 1.1 For any real-valued process S, define the process $S_t^* := \sup_{0 \le r \le t} S_r$ and $A_t := \int_0^t S_r \,\mathrm{d}r$ whenever they exist. Let $\mathbb{F}$ be a filtration satisfying the usual conditions.

(a) Show that S^* and A are both adapted and RCLL if S is adapted and RCLL.

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuous. Suppose that S is adapted and continuous. Show that $\vartheta_t := f(S_t, S_t^*, A_t)$, $t \ge 0$, is then predictable. What happens if S is only RCLL and adapted?

Exercise 1.2 Construct a similar arbitrage strategy as $\mathbb{1}_{[\![0,\tau]\!]}$ from the lecture on a finite horizon and with a positive process.

Exercise 1.3 Let $X = (X_t)_{t \ge 0}$ be a continuous semimartingale defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \ge 0}, P)$. The *local time* of X at 0 is a process defined by

$$L_t^X(0) := |X_t| - |X_0| - \int_0^t \operatorname{sgn}(X_s)\, dX_s,$$

where $\operatorname{sgn}(x) = \mathbb{1}_{(0,\infty)}(x) - \mathbb{1}_{(-\infty,0]}(x)$.

(a) Show that the process $\operatorname{sgn}(X)$ is $\mathbb{F}$-predictable and the process $(L_t^X(0))_{t \ge 0}$ is increasing, nonnegative, and continuous in t.

Hint: First, we can choose (by interpolating quadratically) a family of *convex* C^2-functions f_h such that $f_h(x) = -x$ for $x \le 0$, $f_h(x) = x - h$ for $x \ge h$ and $f_h(x) \to |x|$, $f_h'(x) \to \operatorname{sgn}(x)$ for all x as $h \to 0$. Then apply Itô's formula for each f_h. Finally apply the dominated convergence theorem for stochastic integrals to conclude the proof.

(b) For any $y \in \mathbb{R}$, the *local time* $L^X(y)$ of X at y is defined as the local time of the process $X - y$ at 0, i.e., $L_t^X(y) := |X_t - y| - |X_0 - y| - \int_0^t \operatorname{sgn}(X_s - y)\, dX_s$ for all $t \ge 0$. Let $K \in \mathbb{R}$ be any real number. Prove that

$$(X_t - K)^+ = (X_0 - K)^+ + \int_0^t \mathbb{1}_{\{X_s > K\}}\, dX_s + \frac{1}{2} L_t^X(K),$$

where $(x - K)^+ := \max(x - K, 0)$ for $x \in \mathbb{R}$.

Exercise 1.4 Let the financial market on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, $T < \infty$, be described by a reference asset $S^0 = 1$ and one risky asset S being a geometric Brownian motion, i.e.

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 = s_0 > 0, \tag{1}$$

for some given constants $\mu \in \mathbb{R}, \sigma > 0$. Fix $K > 0$. We start with one share if $S_0 > K$ and with no share if $S_0 \leq K$. Whenever the stock price falls below K (or equals K), the share is sold, and whenever the price returns to a level strictly above K, one share is bought again. Thus, the amount held in the reference asset is given by $\varphi_t^0 = -K\mathbb{1}_{\{S_t>K\}}$, and the amount held in the risky asset is given by $\vartheta_t = \mathbb{1}_{\{S_t>K\}}$.

(a) Verify that the geometric Brownian motion S satisfying (1) has the expression

$$S_t = s_0 \exp\Big(\sigma W_t + (\mu - \sigma^2/2)t\Big).$$

(b) Show that for each $t \in (0, T]$, it holds that

$$P[S_t > K] > 0 \text{ and } P[S_t < K] > 0.$$

(c) Let $L^S(K)$ be the local time of S at K defined as in Exercise 1.3. Show that $P[L_t^S(K) > 0] > 0$ holds for all $t \in (0, T]$.

Hint: Recall that by Girsanov's theorem, there exists a measure Q which is equivalent to P on $\mathcal{F}_T$ and such that $(S_t)_{t\in[0,T]}$ is a martingale with respect to Q. You can take the Q-expectation of $(S_t - K)^+$ and apply Jensen's inequality to get the desired result. The formula in Exercise 1.3, (b) will be very helpful. You may also use the fact that if S is a continuous martingale and H is a bounded predictable process, then the stochastic integral $\int_0^\cdot H\,dS$ is also a continuous martingale.

(d) Use the result in (c) to conclude that the so-called *stop-loss start-gain strategy* (φ^0, ϑ) defined above is not a self-financing strategy.

Hint: Consider the formula in Exercise 1.3, (b).