Mathematical Finance

Exercise sheet 2

Please hand in your solutions by Wednesday, 03.10.2018, 12:00 into your assistant's box next to HG G 53.2.

Exercise 2.1 Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space and the discounted price process $S = (1, S^1)$ is a semimartingale.

(a) Show that NA $\iff \mathcal{C}^0_{\text{adm}} \cap L^0_+ = \{0\}$.

(b) Show that $\mathcal{G} \cap L^0_+ = \{0\} \iff \mathcal{C}^0 \cap L^0_+ = \{0\}$.

(c) Show that NA $\iff (\mathcal{C}^0_{\text{adm}} \cap L^\infty) \cap L^0_+ = \{0\}$.

(d) Show that even if there exists an ELMM Q, NA_{elem} can fail. In particular, can you argue directly that ϑ is not admissible in you example?

Exercise 2.2 Suppose $Q \approx P$ is such that $E_Q[Y] \leq 0$ for all $Y \in \mathcal{C}^0_{\text{adm}} \cap L^\infty$.

(a) Show that $E_Q[G_T(\vartheta)] \leq 0$ for all $\vartheta \in \Theta_{\text{adm}}$.

(b) Suppose we are in discrete time and S is adapted and locally bounded. Show that S is then a local Q-martingale.

Exercise 2.3 Let $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space with time horizon $T > 0$ and $\bar{S} = (1, S^1_t, \ldots, S^d_t)_{t\in[0,T]}$ a general continuous-time financial market.

(a) Show that if there exists an (elementary) arbitrage opportunity

$$\vartheta = \sum_{k=1}^{N} h_k 1_{]\!]\tau_{k-1},\tau_k]\!]} \in \mathbf{b}\mathcal{E},$$

then there also exists a "one-step buy-and-hold" arbitrage opportunity of the form $\vartheta^* = h 1_{]\!]\sigma_0,\sigma_1]\!]} \in \mathbf{b}\mathcal{E}$.

(b) Assume that S is a semimartingale and satisfies NA. Prove that if $\vartheta \in \Theta_{\text{adm}}$ satisfies $G_T(\vartheta) \geq -c$ P-a.s. for some $c \geq 0$, then $G_\cdot(\vartheta) \geq -c$ P-a.s.

Hint: Use that if ϑ is S-integrable and $C \in \mathcal{P}$ is a predictable set, then $1_C\,\vartheta$ is S-integrable, too.

Exercise 2.4 Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions. Set

$$\begin{aligned}
\mathcal{H}^2 &:= \{\text{RCLL martingales } M \text{ such that } \sup_{t\geq 0} E[M_t^2] < \infty\},\\
\mathcal{H}_0^2 &:= \{M \in \mathcal{H}^2 : M_0 = 0\},\\
\mathcal{H}_0^{2,c} &:= \{M \in \mathcal{H}_0^2 : M \text{ is continuous}\},\\
\mathcal{H}_0^{2,d} &:= (\mathcal{H}_0^{2,c})^\perp = \{N \in \mathcal{H}_0^2 : E[M_\infty N_\infty] = 0,\ \forall N \in \mathcal{H}_0^{2,c}\}.
\end{aligned}$$

Recall that $\mathcal{H}^2$ is a Hilbert space with the inner product $(M, N)_{\mathcal{H}^2} = E[M_\infty N_\infty]$. Obviously, $\mathcal{H}_0^2$, $\mathcal{H}_0^{2,c}$, $\mathcal{H}_0^{2,d}$ are all closed subspaces (of $\mathcal{H}^2$). So each $M \in \mathcal{H}^2$ has a unique decomposition $M = M_0 + M^c + M^d$, where $M^c \in \mathcal{H}_0^{2,c}$ and $M^d \in \mathcal{H}_0^{2,d}$. We also use $\mathcal{H}_{0,\text{loc}}^2, \mathcal{H}_{0,\text{loc}}^{2,c}, \mathcal{H}_{0,\text{loc}}^{2,d}$ to denote the space of processes that are locally in $\mathcal{H}_0^2$, $\mathcal{H}_0^{2,c}$, $\mathcal{H}_0^{2,d}$ respectively.

The following two facts can be used freely throughout this exercise:

i) for $M \in \mathcal{H}_0^2$, we have $M \in \mathcal{H}_0^{2,d} \iff E[M_\infty^2] = E[\sum_{s>0}(\triangle M_s)^2]$;

ii) for each $M \in \mathcal{H}_0^2$, there exists a unique adapted, increasing, RCLL process $[M] = ([M]_t)_{t\geq 0}$ null at 0 with $\triangle[M] = (\triangle M)^2$ and such that $M^2 - [M]$ is a uniformly integrable martingale null at 0.

(a) For $L, M \in \mathcal{H}_0^2$, define $[L, M] = \frac{1}{4}([L + M] - [L - M])$. Show that $[L, M]$ is the unique process B, null at 0, of finite variation with $\triangle B = \triangle L \triangle M$ and such that $LM - B$ is a uniformly integrable martingale.

(b) Show that $[\cdot, \cdot]$ is bilinear and symmetric. It can be shown that $[L, M]^\tau = [L, M^\tau]$ for every stopping time τ.

(c) Suppose $L \in \mathcal{H}_0^{2,c}$ and $M \in \mathcal{H}_0^{2,d}$. Prove $[L, M] \equiv 0$.

Hint: Show that LM is a uniformly integrable martingale and that $[L, M]$ is continuous.

(d) Suppose $L, M \in \mathcal{H}_{0,\text{loc}}^2$. Show that

$$[L, M] = \langle L^c, M^c\rangle + [L^d, M^d] = \langle L^c, M^c\rangle + \sum_{0<s\leq \cdot} \triangle L_s \triangle M_s.$$

(e) Let $N = (N_t)_{t\geq 0}$ be a *Poisson process* with rate $\lambda > 0$ and $(Y_k)_{k\geq 1}$ a sequence of random variables independent of N and such that the Y_k are i.i.d., square-integrable with mean μ and $P[Y_k = 0] = 0$. Define the *compensated compound Poisson process* $X = (X_t)_{t\geq 0}$ by

$$X_t := \sum_{k=1}^{N_t} Y_k - \mu\lambda t,$$

and assume about the filtration that X is a Lévy process with respect to $(\mathcal{F}_t)_{t\geq 0}$. (This is for instance satisfied if the filtration is generated by X.) Show that $X \in \mathcal{H}^{2,d}_{0,\mathrm{loc}}$ and $[X]_t = \sum_{k=1}^{N_t} Y_k^2$.

Hint: For $n \in \mathbb{N}$, denote by $\tau_n := \inf\{t \geq 0 : N_t = n\}$ the n-th jump time of the Poisson process. The elementary theory of Poisson processes shows that τ_n is Gamma(n, λ)-distributed. In particular, $E[\tau_n] = \frac{n}{\lambda}$ and $\mathrm{Var}(\tau_n) = \frac{n}{\lambda^2}$, $n \in \mathbb{N}$.