Coordinator
Zhouyi Tan

Mathematical Finance

Exercise sheet 3

Please hand in your solutions by Wednesday, 17.10.2018, 12:00 into your assistant's box next to HG G 53.2.

Exercise 3.1

(a) Show that Brownian motion and the simple Poisson process are good integrators.

(b) Let B be a Brownian motion and $f : \mathbb{R} \to \mathbb{R}$ a function which is continuous except for one point where there is a jump. Show that $X = f(B)$ is **not** a good integrator.

Exercise 3.2 Recall that a process X is of class DL if for each $t \geq 0$, the family $\{X_\tau : \tau \leq t \text{ stopping time}\}$ is uniformly integrable.

(a) Show that X is a supermartingale if X is nonnegative and locally a supermartingale with integrable X_0.

(b) Suppose X is nonnegative and locally a submartignale. Show that X is a submartingale iff X is of class DL.

(c) Suppose X is locally a supermartingale. Show that X is a supermartingale iff X_0 is integrable and X^- is of class DL.

Exercise 3.3

(a) Suppose that $f, g : [0,T] \to \mathbb{R}$ are functions of finite variation. Establish the following integration by parts formula

$$f(T)g(T) - f(0)g(0) = \int_0^T f(s-)\,\mathrm{d}g(s) + \int_0^T g(s)\,\mathrm{d}f(s).$$

Written symmetrically, we have

$$f(T)g(T) - f(0)g(0) = \int_0^T f(s-)\,\mathrm{d}g(s) + \int_0^T g(s-)\,\mathrm{d}f(s) + \sum_{0<s\leq T} \triangle f(s) \triangle g(s).$$

(b) For $t > 0$, define $\mathcal{F}_{t-} = \sigma(\bigcup_{s<t} \mathcal{F}_s)$. Show that if X is predictable, then X_t is $\mathcal{F}_{t-}$-measurable for each $t > 0$.

(c) Suppose that M is a bounded martingale and A is an increasing integrable RCLL predictable process null at 0. Show that $E[M_T A_T] = E[\int_0^T M_{s-}\,\mathrm{d}A_s]$.

Exercise 3.4 Recall that the space $\mathcal{H}_0^1$ (on the time interval $[0,T]$) is defined by

$$\mathcal{H}_0^1 := \Big\{ M \in \mathcal{M}_{loc} : M_0 = 0, M_T^* := \sup_{0 \leq t \leq T} |M_t| \in L^1 \Big\}.$$

(a) Show that every local martingale M in $\mathcal{H}_0^1$ is of class D; that is, the family $\{M_\tau : \tau \text{ stopping time}\}$ is uniformly integrable.

(b) Show that every local martingale in $\mathcal{H}_0^1$ is a martingale.

(c) Show that every local martingale is locally in $\mathcal{H}_0^1$.

Exercise 3.5 Let $X = (X_t)_{0 \leq t \leq T}$ be an adapted RCLL process and $\mathcal{D}_n := k2^{-n}T,\ k := 0, 1, ..., 2^n$, the n-th dyadic partition of $[0,T]$, for each $n \in \mathbb{N}$. Suppose that X is bounded.

(a) Show that for each stopping time ρ with values in $[0,T]$, we have

$$|\mathrm{MV}(X^\rho, \mathcal{D}_n) - \mathrm{MV}(X^{\rho+}, \mathcal{D}_n)| \leq 2\|X\|_\infty.$$

(b) Show that $\mathrm{MV}(X) = \lim_{n\to\infty} \mathrm{MV}(X, \mathcal{D}_n)$.