Coordinator
Zhouyi Tan

Mathematical Finance

Exercise sheet 4

Please hand in your solutions by Wednesday, 31.10.2018, 12:00 into your assistant's box next to HG G 53.2.

Exercise 4.1

(a) Show that $d_0(Z^1, Z^2) := E[1 \wedge |Z^1 - Z^2|]$ is a metric on L^0 and that $Z_n \to Z$ in probability iff $d_0(Z_n, Z) \to 0$. So d_0 metrizes the convergence in probability.

(b) Recall from the lecture $\mathbb{L}$ (resp. $\mathbb{D}$) of adapted LCRL (resp. RCLL) processes. Define

$$d(X^1, X^2) := E[1 \wedge (X^1 - X^2)^*_T] = E\Big[1 \wedge \sup_{0 \le s \le T} |X^1_s - X^2_s|\Big]$$

on $\mathbb{L}$ (resp. $\mathbb{D}$). Assume that the filtration $\mathbb{F}$ is complete. Show that both $(\mathbb{L}, d)$ and $(\mathbb{D}, d)$ are complete metric spaces.

(c) Show that $b\mathcal{E}_0$ is dense in $\mathbb{L}$ for d.

Exercise 4.2

(a) Show that a nonemepty subset $C \subseteq L^0$ is bounded in L^0 iff for every sequence $\lambda_n \downarrow 0$ in $(0, \infty)$, $\lambda_n c_n \to 0$ in L^0 for every sequence $(c_n)_{n \in \mathbb{N}} \subseteq C$.

(b) Show that an adapted RCLL process X is a good integrator iff the set $I_X(H) = \{H \bullet X_T : H \in b\mathcal{E}, \|H\|_\infty \le 1\}$ is bounded in L^0.

Exercise 4.3

(a) Let $(X^n)_{n \in \mathbb{N}} \subseteq \mathbb{D}$ be a sequence of local martingales converging to a limit X under the metric d. Suppose that $\sup_n |\triangle X^n|$ is locally integrable. Show that X is a local martingale.

(b) Suppose that $M \in \mathcal{M}_{0,\mathrm{loc}}$ and H is predictable and locally bounded. Show that $H \bullet M \in \mathcal{M}_{0,\mathrm{loc}}$.

Exercise 4.4 The goal of this exercise is to establish some basic properties of optional quadratic variations. For two semimartingales X, Y, use polarization to define $[X, Y] = \frac{1}{4}([X + Y] - [X - Y])$.

(a) Establish the integration by parts formula

$$XY = X_0Y_0 + \int X_- \,\mathrm{d}Y + \int Y_- \,\mathrm{d}X + [X,Y].$$

(b) Show that $\triangle[X,Y] = \triangle X \triangle Y$. In particular, $\triangle[X] = (\triangle X)^2$.

(c) Show that $\sum_{s\leq t}(\triangle X_s)^2 < \infty$ P-a.s. for all $t \in [0,T]$.

(d) Suppose X is a semimartingale and V is an FV process. Compute $[X,V]$.

Exercise 4.5

(a) Suppose that A is of FV and $(H^n)_{n\in\mathbb{N}} \subseteq b\mathcal{P}$ satisfies $\|H^n\|_\infty \leq 1$ and $H^n \to 0$ pointwise. Show that $d'_E(H^n \bullet A, 0) \to 0$.

(b) For $M \in \mathcal{M}_{0,\mathrm{loc}}$ and $(H^n)_{n\in\mathbb{N}} \subseteq b\mathcal{P}$ with $\|H^n\|_\infty \leq \|H\|_\infty$ $\forall n$, for $H \in b\mathcal{P}$ and $H^n \to H$ in $L^1(M)$ or pointwise, prove that $(H^n \bullet M)_{n\in\mathbb{N}}$ is Cauchy in $(\mathcal{S}, d'_E)$.

(c) For $S \in \mathcal{S}$ and $H \in b\mathcal{P}_{\mathrm{loc}}$, construct $H \bullet S$.