Mathematical Finance

Exercise sheet 5

Please hand in your solutions by Wednesday, 07.11.2018, 12:00 into your assistant’s box next to HG G 53.2.

Exercise 5.1 Construct an example where $S$ is a martingale but $G_{adm} = \{0\}$.

Exercise 5.2 Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ be a filtered probability space, $S = (S^1, \ldots, S^d)_{t \in [0,T]}$ a $d$-dimensional semimartingale and $Q \approx P$ on $\mathcal{F}_T$ an equivalent probability measure.

(a) Assume that $\mathcal{F}_0$ is trivial and $Q$ is a separating measure for $S$. Show that if $S$ is (locally) bounded, then $Q$ is an equivalent (local) martingale measure for $S$.

(b) Assume that $Q$ is an equivalent $\sigma$-martingale measure for $S$. Show that it is also an equivalent separating measure.

(c) Now assume that $d = 1$, that $(\mathcal{F}_t)_{t \in [0,T]}$ is the natural (completed) filtration of $S$ and that the process $S = (S_t)_{t \in [0,T]}$ is of the form

$$S_t = \begin{cases} 0 & \text{for } 0 \leq t < T, \\ X & \text{for } t = T, \end{cases}$$

where $X$ is normally distributed with mean $\mu \neq 0$ and variance $\sigma^2 > 0$. Show that in this case, the class $M_{sep}$ of equivalent separating measures for $S$ is strictly bigger than the class $M_{\sigma}$ of equivalent $\sigma$-martingale measures for $S$.

Exercise 5.3 Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space and $X = (X_t)_{t \geq 0}$ a $d$-dimensional RCLL $\sigma$-martingale with $X_0 = 0$.

(a) Prove that there exists a sequence of predictable sets $(\Sigma_n)_{n \geq 1}$ (i.e. each $\Sigma_n$ lies in the predictable $\sigma$-algebra $\mathcal{P}$ on $\Omega \times [0, \infty)$) such that $\Sigma_n \subseteq \Sigma_{n+1}$ for all $n$, $\bigcup_n \Sigma_n = \Omega \times [0, \infty)$ and for each $n$, the process $1_{\Sigma_n} \cdot X = (1_{\Sigma_n} \cdot X^i)_{i = 1, \ldots, d}$ is a $d$-dimensional uniformly integrable martingale.

(b) Show that every continuous $\sigma$-martingale is a local martingale.