Mathematical Finance

Exercise sheet 6

Please hand in your solutions by Wednesday, 14.11.2018, 12:00 into your assistant's box next to HG G 53.2.

Exercise 6.1 The general Itô formula says that if X is a semimartingale with X, X_- taking values in an open set $U \subseteq \mathbb{R}$ and $f : U \to \mathbb{R}$ is a twice differentiable function, then $f(X)$ is again a semimartingale and

$$\begin{aligned} f(X_t) = f(X_0) &+ \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f''(X_{s-})d[X]_s \\ &+ \sum_{0<s\leq t}\left(\triangle f(X_s) - f'(X_{s-})\triangle X_s - \frac{1}{2}f''(X_{s-})\triangle X_s^2\right). \end{aligned}$$

(a) Show that if $Y > 0$ is a semimartingale with $Y_- > 0$, then $1/Y$ is also a semimartingale. Where exactly do you use $Y_- > 0$?

(b) Show that there is at most one numéraire portfolio.

Exercise 6.2

(a) Let Z be a strictly positive local martingale with $Z_0 = 1$ and S an adapted, continuous process with $S_0 = 0$ such that ZS is a σ-martingale. Prove that ZS is a local martingale.

(b) Show that if S admits a P-equivalent σ-martingale density and $Q \approx P$ on $\mathcal{F}_T$, then S also admits a Q-equivalent σ-martingale density.

Hint: You can use the fact that if M is a local martingale and H is predictable and locally bounded, then $H \bullet M$ is a local martingale.

(c) Show that for any EσMD Z for S and any $X \in \mathcal{X}^1$, the product ZX is a P-supermartingale.

Exercise 6.3 Consider a stock price model based on a 2-dimensional Brownian motion (W, W') in the latter's own filtration, where the 1-dimensional (discounted) stock price follows

$$dS_t = \mu_t dt + \sigma_t dW_t$$

and $\mu_t = \mu(t, S_t, Y_t)$ and $\sigma_t = \sigma(t, S_t, Y_t) > 0$ are determined by continuous functions depending on the diffusion Y which follows

$$\begin{aligned} dY_t &= b(t, Y_t)dt + a(t, Y_t)dB_t, \\ Y_0 &= y_0. \end{aligned}$$

Here B is a Brownian motion correlated to W, defined by

$$B_t = \rho W_t + \sqrt{1-\rho^2}\, W'_t$$

for some constant $\rho \in (0,1)$. We assume that $\sigma > 0$ and that the function $\frac{\mu}{\sigma}$ is uniformly bounded. The process Y is often called *stochastic factor* in this context.

(a) Show that $d\langle B, W\rangle_t = \rho dt$ and $d\langle S, Y\rangle_t = a(t, Y_t)\sigma(t, S_t, Y_t)\rho dt$.

(b) What is the general form of the density process of an ELMM Q for S?

(c) Find the dynamics of S and Y under such a measure; that is, give stochastic differential equations for these processes involving only Brownian motions under Q, not under P.