Mathematical Finance

Exercise sheet 6

Please hand in your solutions by Wednesday, 14.11.2018, 12:00 into your assistant’s box next to HG G 53.2.

Exercise 6.1 The general Itô formula says that if $X$ is a semimartingale with $X, X_-$ taking values in an open set $U \subseteq \mathbb{R}$ and $f : U \rightarrow \mathbb{R}$ is a twice differentiable function, then $f(X)$ is again a semimartingale and

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d[X]_s + \sum_{0 < s \leq t} \left( \Delta f(X_s) - f'(X_{s-}) \Delta X_s - \frac{1}{2} f''(X_{s-}) \Delta X_s^2 \right).$$

(a) Show that if $Y > 0$ is a semimartingale with $Y_- > 0$, then $1/Y$ is also a semimartingale. Where exactly do you use $Y_- > 0$?

(b) Show that there is at most one numéraire portfolio.

Exercise 6.2

(a) Let $Z$ be a strictly positive local martingale with $Z_0 = 1$ and $S$ an adapted, continuous process with $S_0 = 0$ such that $ZS$ is a $\sigma$-martingale. Prove that $ZS$ is a local martingale.

(b) Show that if $S$ admits a $P$-equivalent $\sigma$-martingale density and $Q \approx P$ on $\mathcal{F}_T$, then $S$ also admits a $Q$-equivalent $\sigma$-martingale density.

Hint: You can use the fact that if $M$ is a local martingale and $H$ is predictable and locally bounded, then $H \cdot M$ is a local martingale.

(c) Show that for any $\mathbb{E}\sigma$MD $Z$ for $S$ and any $X \in \mathcal{X}^1$, the product $ZX$ is a $P$-supermartingale.

Exercise 6.3 Consider a stock price model based on a 2-dimensional Brownian motion $(W, W')$ in the latter’s own filtration, where the 1-dimensional (discounted) stock price follows

$$dS_t = \mu_t dt + \sigma_t dW_t$$

and $\mu_t = \mu(t, S_t, Y_t)$ and $\sigma_t = \sigma(t, S_t, Y_t) > 0$ are determined by continuous functions depending on the diffusion $Y$ which follows

$$dY_t = b(t, Y_t) dt + a(t, Y_t) dB_t,$$

$Y_0 = y_0$. 

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Here $B$ is a Brownian motion correlated to $W$, defined by

$$B_t = \rho W_t + \sqrt{1 - \rho^2} W'_t$$

for some constant $\rho \in (0, 1)$. We assume that $\sigma > 0$ and that the function $\frac{\mu}{\sigma}$ is uniformly bounded. The process $Y$ is often called *stochastic factor* in this context.

(a) Show that $d \langle B, W \rangle_t = \rho dt$ and $d \langle S, Y \rangle_t = a(t, Y_t) \sigma(t, S_t, Y_t) \rho dt$.

(b) What is the general form of the density process of an ELMM $Q$ for $S$?

(c) Find the dynamics of $S$ and $Y$ under such a measure; that is, give stochastic differential equations for these processes involving only Brownian motions under $Q$, not under $P$. 