Mathematical Finance

Exercise sheet 7

Please hand in your solutions by Wednesday, 05.12.2018, 12:00 into your assistant's box next to HG G 53.2.

Exercise 7.1 Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space satisfying the usual conditions. Moreover, let $X = (X_t)_{t\in[0,T]}$ be a a compound Poisson process with jump intensity $\lambda > 0$ and jump distribution ν, i.e., $X_t := \sum_{k=1}^{N_t} Y_k$, where $N = (N_t)_{t\in[0,T]}$ is a Poisson process with rate λ and $(Y_k)_{k\in\mathbb{N}}$ a sequence of random variables independent of N such that the Y_k are i.i.d. with distribution ν (with $\nu(\{0\}) = 0$). Suppose that $(\mathcal{F}_t)_{t\geq 0}$ is such that X is a Lévy process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. Let $\widetilde{\lambda} > 0$ and let $\widetilde{\nu} \approx \nu$ be an equivalent probability measure on $\mathbb{R}$. Define the exponential Lévy process $Z = (Z_t)_{t\in[0,T]}$ by

$$Z_t := \exp\left(\sum_{k=1}^{N_t} \phi(Y_k) + (\lambda - \widetilde{\lambda})t\right),$$

where $\phi = \log\left(\frac{\widetilde{\lambda}}{\lambda}\frac{d\widetilde{\nu}}{d\nu}\right)$.

(a) Show that Z is a P-martingale.

(b) Define the probability measure $Q \approx P$ on $\mathcal{F}_T$ by $dQ = Z_T\, dP$. Show that under Q, X is again a compound Poisson process for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ with rate $\widetilde{\lambda}$ and jump distribution $\widetilde{\nu}$.

Hint: Show that X is a Lévy process under Q for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, and calculate the characteristic function of X_1 under Q to determine its law (assuming without loss of generality that $T \geq 1$).

Exercise 7.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space supporting a Poisson process $N = (N_t)_{t\in[0,T]}$ with rate $\lambda > 0$. Denote by $(\mathcal{F}_t^N)_{t\in[0,T]}$ the natural (completed) filtration of N. Define the process $S = (S_t)_{t\in[0,T]}$ by $dS_t = S_{t-}(\mu\, dt + \frac{\sigma}{\sqrt{\lambda}}\, d\widetilde{N}_t)$, $S_0 = s_0 > 0$, where $\mu \in \mathbb{R}$, $\sigma > 0$ and $\widetilde{N}_t = N_t - \lambda t$, $t \geq 0$, denotes the compensated Poisson process. Assume that S satisfies NFLVR. Moreover, use without proof that the equivalent martingale measure Q^λ for S is unique. For $\rho > 0$ denote by $\overline{\Psi}_\rho$ the tail distribution function of a Poisson random variable with parameter ρ, i.e., $\overline{\Psi}_\rho(x) := P[X_\rho > x]$, where X_ρ has a Poisson distribution with parameter ρ.

(a) Construct explicitly the EMM Q^λ.

Hint: You may use the remark after Exercise E.3.

(b) Show that the risk-neutral price of a *cash-or-nothing call option* with payoff $H = 1_{\{S_T > K\}}$ with maturity T and strike $K > 0$ is given by

$$\overline{\Psi}_{\left(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}\right)T}\left(\frac{\log\frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right)T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right).$$

Hint: Use that for a Lévy process R with triplet (a, σ^2, ν), the stochastic exponential of R is given by

$$\mathcal{E}(R)_t := \exp\left(R_t - \frac{1}{2}\sigma^2 t\right)\prod_{0<s\le t}(1 + \Delta R_s)\exp(-\Delta R_s).$$

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(c) Show that the risk-neutral price of a *stock-or-nothing call option* with payoff $H = S_T 1_{\{S_T > K\}}$ with maturity T and strike $K > 0$ is given by

$$S_0\overline{\Psi}_{\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)\left(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda}\right)T}\left(\frac{\log\frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right)T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right).$$

Hint: Define the measure $\widetilde{Q}^\lambda \approx Q^\lambda$ on $\mathcal{F}_T$ by $\frac{d\widetilde{Q}^\lambda}{dQ^\lambda} := S_T/S_0$, and work under this measure.

(d) Derive the risk-neutral price C_0^λ of a call option with payoff $H = (S_T - K)^+$ with maturity T and strike K. Moreover, show that

$$\lim_{\lambda\to\infty} C_0^\lambda = S_0\Phi\left(\frac{\log\frac{S_0}{K} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{S_0}{K} - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right),$$

where Φ denotes the distribution function of a standard normal random variable. This means that for large λ, the arbitrage-free price in the Poisson model is very close to the Black–Scholes price with the same parameter σ.

Hint: Use that if X_ρ has a Poisson distribution with parameter ρ, then $\frac{X_\rho - \rho}{\sqrt{\rho}}$ converges weakly to a standard normal random variable for $\rho \to \infty$. Moreover, use the fact that if $(F_n)_{n\in\mathbb{N}}$ is a sequence of distribution functions converging *pointwise* to a *continuous* distribution function F, then the convergence is also *uniform*.

Exercise 7.3 In the Black–Scholes model, prove that the so-called *digital option* with undiscounted payoff $\tilde{H} = 1_{\{\tilde{S}_T > \tilde{K}\}}$, $\tilde{K} > 0$, is attainable, and calculate its arbitrage-free price process and the replicating strategy.

Exercise 7.4 Consider the one-period trinomial model. This means $T = 1$, a constant bank account $B = 1$, and the price process $(S_t)_{t\in\{0,1\}}$ has a constant $S_0 > 0$ and $S_1 = S_0 Z$ with Z taking three possible values $1 + u > 1 + m > 1 + d$ under a given measure P. Moreover, assume $\Omega = \{\omega_u, \omega_m, \omega_d\}$, where $\omega_u = \{Z = 1 + u\}$, $\omega_m = \{Z = 1 + m\}$ and $\omega_d = \{Z = 1 + d\}$. Furthermore, let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 = \sigma(S_1)$. For a probability measure Q, write $q_i = Q[\{\omega_i\}], i = u, m, d$. Assume that $P[\{\omega\}] > 0$ for all $\omega \in \Omega$.

(a) For which $u > m > d$ is there arbitrage? In the case where NA holds true, characterize all equivalent local martingale measures in terms of q_u, q_m, q_d.

(b) Let $H = (S_1 - K)^+$ be a call option with strike $K > 0$ satisfying $S_0(1 + m) > K > S_0(1 + d)$ and $S_0 \neq K$. Show that H is not attainable.

(c) Assume that $u > 0 > d$. Let $H = (S_1 - K)^+$ be a call option with strike $K > 0$. Compute its superreplication price $\pi^s(H)$. Moreover, prove that there exists a martingale measure Q_a *absolutely continuous* with respect to P such that $\pi^s(H) = E_{Q_a}[H]$.

Exercise 7.5 Consider a market in finite discrete time with $\mathcal{F}_0$ being trivial.

(a) Prove or disprove:

1. $(S, \mathbb{F})$ is complete implies that $\#(\mathbb{P}_e) \leq 1$.
2. $\#(\mathbb{P}_{\mathrm{e}}) \leq 1$ implies that $(S, \mathbb{F})$ is complete.

(b) Prove that if NA holds, then $\inf_{Q\in\mathbb{P}_{\mathrm{e}}(S)} E_Q[H] < \infty$ for any $H \in L^0_+(\mathcal{F}_T)$.

Exercise 7.6 We consider a generalized Black-Scholes model with a bank account $B \equiv 1$ and a stock

$$\frac{dS_t}{S_t} = b(\omega')\,dt + \sigma(\omega')\,dW_t, \quad S_0 > 0 \text{ deterministic.}$$

Here W is a Brownian motion under P in the filtration $\mathbb{F}$ and b, $\sigma > 0$ are random variables which are P-independent of W.

(a) Construct a minimal probability space for such a model.

(b) Describe the set $\mathbb{P}_{\mathrm{e,loc}}$ of equivalent local martingale measures for S in (a).

(c) Let $H = (S_T - K)^+$ be a (discounted) call option. Recall from the lecture the function $u_{BS}(t, x, v)$ for the (discounted) price of the call at time t for known and fixed volatility v and current stock price $x = S_t$. Prove that for each $Q \in \mathbb{P}_{\mathrm{e,loc}}$, we have

$$E_Q[H] = E_Q[u_{BS}(0, S_0, \sigma)] = \int_0^\infty u_{BS}(0, S_0, v)\,\nu_\sigma^Q(dv),$$

where ν_σ^Q is the distribution of σ under Q.

(d) Assume that σ is neither bounded away from zero nor from infinity. Show that

$$\pi_s(H) = S_0 \quad \text{and} \quad \pi_b(H) = (S_0 - K)^+.$$

What does this mean for the superreplicating (subreplicating) strategies?

Hint: Recall

$$u_{BS}(0, S_0, v) = S_0\Phi(d_+) - K\Phi(d_-), \quad d_\pm := \frac{1}{v\sqrt{T}}\Big\{ \log(S_0/K) \pm \frac{1}{2}v^2 T \Big\}.$$

Remark: An analogous result is true in a fairly large class of stochastic volatility models.