Coordinator
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Mathematical Finance

Exercise sheet 8

Please hand in your solutions by Wednesday, 19.12.2018, 12:00 into your assistant's box next to HG G 53.2.

Exercise 8.1 Consider a financial market modeled by a semimartingale S satisfying $\mathbb{P}_{e,\sigma}(S) \neq \varnothing$.

(a) A mapping $\rho : L^\infty \to \mathbb{R}$ is said to be a **coherent risk measure** if it satisfies

i) $H \leq H'$ P-a.s. implies $\rho(H) \geq \rho(H')$;

ii) $\rho(H + c) = \rho(H) - c$ for all $c \in \mathbb{R}$;

iii) $\rho(\lambda H) \leq \lambda \rho(H)$ for all $\lambda \geq 0$;

Show that $\rho := -\pi^s$ is a coherent risk measure.

Hint: Use only the definition of π^s.

(b) i) Show that S satisfies NA iff $g := 0$ is maximal in $\mathcal{G}_{\mathrm{adm}}$.

ii) Suppose that $S = (S_k)_{k=0,1,\ldots,T}$ is a discrete-time process with $T < \infty$. Denote by Θ the space of all predictable processes. Show that if S satisfies NA, then neither $\mathcal{G}_{\mathrm{adm}}$ nor $G_T(\Theta)$ contains any non-maximal element.

Exercise 8.2 Let $U : (0, \infty) \to \mathbb{R}$ be concave and increasing. For $x > 0$, define $u(x) := \sup_{V \in \mathcal{V}(x)} E[U(V_T)]$, where $\mathcal{V}(x) := \{x + \vartheta \bullet S : \vartheta \in \Theta^x_{\mathrm{adm}}\}$.

(a) Show that u is concave and increasing.

(b) If additionally $u(x_0) < \infty$ for some $x_0 > 0$, show that $u(x) < \infty$ for all $x > 0$.

(c) If U is strictly increasing with $U(\infty) < \infty$ and S satisfies NFLVR, then $u(x) < U(\infty)$ for all $x > 0$.

Exercise 8.3 Consider a financial market modeled by an $\mathbb{R}^d$-valued semimartingale S satisfying NFLVR. Let $U : (0, \infty) \to \mathbb{R}$ be a utility function such that $u(x) < \infty$ for some $x \in (0, \infty)$.

(a) Assume that the market is complete in the sense that there exists a unique EσMM Q on $\mathcal{F}_T$. Fix $z > 0$. Show that $h \leq z\frac{\mathrm{d}Q}{\mathrm{d}P}$ P-a.s. for all $h \in \mathcal{D}(z)$. Deduce that

$$j(z) = E\left[J\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right].$$

(b) Consider the Black–Scholes market $(\tilde{S}^0, \tilde{S}^1)$ given by

$$\begin{aligned} \mathrm{d}\tilde{S}_0^0 &= r\tilde{S}_t^0\,\mathrm{d}t, \quad \tilde{S}_0^0 = 1, \\ \mathrm{d}\tilde{S}_t^1 &= \tilde{S}_t^1(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t), \quad \tilde{S}_0^1 = s > 0. \end{aligned}$$

Denote by S^1 the discounted stock price. Let $U : (0, \infty) \to \mathbb{R}$ be defined by $U(x) := \frac{1}{\gamma}x^\gamma$, where $\gamma \in (-\infty, 1) \setminus \{0\}$. Show that for $z > 0$,

$$j(z) = \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\gamma}{(1-\gamma)^2}\frac{(\mu - r)^2 T}{\sigma^2}\right).$$

(c) Show in general that

$$j(z) = \inf_{Q \in \mathbb{P}_{e,\sigma}} E\left[J\left(z\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right].$$

Hint: Modify the argument in (a) by showing that for any $h \in \mathcal{D}(z)$, we have $h \leq z\frac{\mathrm{d}Q}{\mathrm{d}P}$ for some $Q \in \mathbb{P}_{e,\sigma}$.