Mathematical Finance

Exercise sheet 9

Exercise 9.1 Let U be a utility function on $(0, \infty)$ satisfying theInada conditions. Its asymptotic elasticity at $+\infty$ is

$$AE_{+\infty}(U) = \limsup_{x\to\infty} \frac{xU'(x)}{U(x)}.$$

(a) Prove that $AE_{+\infty}(U) \leq 1$.

(b) Assume $U(\infty) > 0$. Show that

$$AE_{+\infty}(U) = \inf\{\gamma > 0 : \exists x_0 \text{ s.t. } U(\lambda x) < \lambda^\gamma U(x)\ \forall \lambda > 1, x \geq x_0\}.$$

(c) Prove that $AE_{+\infty}(U) < 1$ implies both $u'(\infty) = 0$ and $AE_{+\infty}(u) < 1$.

Hint: You may use the fact that two concave functions f, g with $f(\infty) > 0, g(\infty) > 0$ have $AE_+(f) = AE_+(g)$ if there exists $C > 0$ such that $g(x) - C \leq f(x) \leq g(x) + C$ for large x.

Exercise 9.2 For each n, set $J_n(z) := \sup_{x\in(0,n]}(U(x) - xz)$ for $z > 0$. Prove that

$$E[J_n(h)] = \sup_{f\in\mathcal{B}_n} E[U(f) - fh].$$

Exercise 9.3 Let S be an $\mathbb{R}$-valued semimartingale of the form

$$S = S_0 + M + \int \lambda \, \mathrm{d}\langle M\rangle,$$

where M is a continuous local martingale null at 0 and λ is a predictable process such that the mean-variance tradeoff process $K = \int \lambda_s^2 \, \mathrm{d}\langle M\rangle_s$ is bounded. Define $\hat{Z} := \mathcal{E}(-\lambda \bullet M)$ and $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \hat{Z}_T$.

(a) Show that $\hat{P} \in \mathbb{P}_{e,loc}(S)$.

(b) Show that both $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P}$ and $\frac{\mathrm{d}P}{\mathrm{d}\hat{P}}$ have moments of all orders.