Mathematical Finance

Exercise sheet E

Exercise E.1 Let A be an RCLL adapted process of FV null at 0.

(a) Suppose that the variation of A is locally integrable. Show that there exists a unique predictable RCLL FV process A^p null at 0 such that $A - A^p$ is a local martingale.

Remark: The process A^p is called the **compensator** of A.

(b) Show that any local martingale of FV has a locally integrable variation.

Remark: Also, it can be shown that any RCLL predictable process of FV is locally bounded.

Exercise E.2 Let M, N be two local martingales null at 0 with the property that $[M, N]$ is locally integrable.

(a) Show that $\langle M, N \rangle := [M, N]^p$ is the unique predictable RCLL FV process C null at 0 such that $MN - C$ is a local martingale.

(b) Suppose that both M, N are locally in $\mathcal{H}^2$. Can you give a different construction of $\langle M, N \rangle$ in this case? Show that $\langle M, N \rangle$ is the compensator of $[M, N]$.

(c) Show that $M \in \mathcal{H}^2_{0,\mathrm{loc}}$ if and only if $[M]$ has locally integrable variation.

(d) Suppose X is adapted RCLL and of FV. Show that $[X]_t = \sum_{0<s\leq t} (\triangle X_s)^2$.

(e) For a Poisson process with intensity λ, define $M_t := N_t - \lambda t$. Show that $M \in \mathcal{M}^2_{0,\mathrm{loc}}$ and compute $[M]$ and $\langle M \rangle$.

Exercise E.3 Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space satisfying the usual conditions. Moreover, let $R = (R_t)_{t\in[0,T]}$ be a *simple jump-diffusion*, i.e., there exist a Brownian motion $W = (W_t)_{t\in[0,T]}$ and an independent compound Poisson process $X = (X_t)_{t\geq 0}$ with jump intensity $\lambda > 0$ and jump distribution ν (with $\nu(\{0\}) = 0$) such that $R_t = at + \sigma W_t + X_t$, $t \in [0, T]$, where $a \in \mathbb{R}$ and $\sigma \geq 0$. Suppose about the filtration that R is a Lévy process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, and suppose about ν that $\nu((-\infty, -1]) = 0$, i.e. the jumps of R are strictly greater than -1. Define the process $S = (S_t)_{t\in[0,T]}$ by $dS_t := S_{t-}\, dR_t$, $S_0 = s_0 > 0$, i.e., $S = s_0 \mathcal{E}(R)$.

Suppose that R is a martingale for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Show that S is then also a martingale (and not only a local martingale) for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

Remark: It can be shown that in general S satisfies NA if and only if the paths of R are not monotone.